METHOD OF MOMENTS ESTIMATOR (MM)

- always CONSISTENT
- <u>not</u> always ASY EFFICIENT

equates sample moments with theoretical moments (population)

moments we use: means, variances, covariances.

for the linear model:

$$y_t = \beta_1 + \beta_2 x_t + e_t$$

under assumptions:

1) $E(e_t) = 0$

2) $E(x_t e_t) = 0$ covar btw error and random regressor is (contemporaneous) 0

let $\hat{\beta}_1$ and $\hat{\beta}_2$ be MME of β_1 and β_2.

Let $\hat{e}_t = y_t - \hat{\beta}_1 - \hat{\beta}_2 x_t$

assumptions 1) and 2) are assumptions on MOMENTS.

THEIR SAMPLE COUNTERPARTS ARE:

1) $\frac{1}{T}\sum \hat{e}_t = 0$

2) $\frac{1}{T}\sum x_t \hat{e}_t = 0$

(by the law of large numbers, the sample mean converges to the expected value)

more explicitly:

1) $\frac{1}{T}\sum \hat{e}_t = \frac{1}{T}\sum\left(y_t - \hat{\beta}_1 - \hat{\beta}_2 x_t\right) = \frac{1}{T}\left(\sum y_t - T\hat{\beta}_1 - \hat{\beta}_2 \sum x_t\right)$

$\frac{1}{T}\sum \hat{e}_t = 0 \Rightarrow \sum y_t - T\hat{\beta}_1 - \hat{\beta}_2 \sum x_t = 0$

2) $\frac{1}{T}\sum x_t \hat{e}_t = \frac{1}{T}\sum\left(x_t y_t - \hat{\beta}_1 x_t - \hat{\beta}_2 x_t^2\right)$

$\frac{1}{T}\sum x_t \hat{e}_t = 0 \Rightarrow \sum x_t y_t - \hat{\beta}_1 \sum x_t - \hat{\beta}_2 \sum x_t^2 = 0$

rearranging terms

$$T\hat{\beta}_1 + \sum x_t \hat{\beta}_2 = \sum y_t$$
$$\sum x_t \hat{\beta}_1 + \sum x_t^2 \hat{\beta}_2 = \sum x_t y_t$$

we obtain the normal equations, i.e. those that lead directly to the OLS estimator. Hence in this model the expression of the MM coincides with that of OLS, BUT HAS AN ASYMPTOTIC JUSTIFICATION

MEASUREMENT ERROR PROBLEM.

data measured with error:

self-esteem, GNP, ...

We distinguish the: observed variables and the unobserved var (latent var.). We don't know their values as we observe them with an error.

let z_t be observations on the latent z_t^*

let y_t y_t^*

$$z_t = z_t^* + u_t$$

$$y_t = y_t^* + v_t$$

where u, v: measurement errors.

Assumptions:

$$E(u_t) = 0 \quad , \quad E(v_t) = 0$$

$$\text{var}(u_t) = E(u_t^2) = \sigma_u^2 \qquad \text{var}(v_t) = E(v_t^2) = \sigma_v^2$$

$$\text{cov}(u_t, u_s) = E(u_t u_s) = 0 \qquad \text{cov}(v_t, v_s) = E(v_t v_s) = 0 \quad , (t \neq s)$$

$$\text{cov}(u_t, v_s) = E(u_t v_s) = 0 \quad \Rightarrow \text{cov}(u_t, u_s) = 0$$

for all t, s

- u_t, v_t are INDEPENDENT OF z_t^*, y_t^*

$\Rightarrow E(u_t z_t^*) = E(u_t) E(z_t^*)$ for example.

THE econometrician would like to estimate

$$y_t^* = \beta_1 + \beta_2 z_t^*$$

BUT

y_t^*, z_t^* LATENT, ONLY y_t, z_t are available

$y_t = y_t^* + v_t \Rightarrow y_t^* = y_t - v_t$

$z_t = z_t^* + u_t \Rightarrow z_t^* = z_t - u_t$

$$(y_t - v_t) = \beta_1 + \beta_2 (z_t - u_t)$$

$$y_t = \beta_1 + \beta_2 z_t + \underbrace{v_t - \beta_2 u_t}$$

MODEL: $\boxed{y_t = \beta_1 + \beta_2 z_t + e_t}$

where $e_t = v_t - \beta_2 u_t$.

WE CHECK IF WE CAN USE OLS:

1) $E(e_t) = E(v_t) - \beta_2 E(u_t) = 0$

2) $\text{var}(e_t) = \text{var}(v_t) + \beta_2^2 \text{var}(u_t) - 2 \cdot \beta_2 \cdot \text{cov}(u_t, v_t)$

$= \sigma_v^2 + \beta_2^2 \sigma_u^2$

What about correlation btw regressors and errors?
What for ex. can we say about z_t and e_t? in the model?

$$cov(z_t, e_t) = E\Big(\big(z_t - E(z_t)\big)\cdot\big(e_t - E(e_t)\big)\Big)$$

$$= E\Big(\big(z_t - z_t^*\big)\cdot e_t\Big)$$

because $E(z_t) = E(z_t^* + u_t) = E(z_t^*) + \underset{\substack{\| \\ 0}}{E(u_t)} = z_t^*$

$$= E\Big(u_t\big(v_t - u_t\beta_2\big)\Big)$$

$$= E\big(u_t v_t\big) - E\big(u_t u_t \beta_2\big)$$

$$= 0 - E(u_t^2)\cdot\beta_2 =$$

$$= -\beta_2\cdot\sigma_u^2$$

next step: check for consistency of the OLS estimator.
To do it recall that in the model

$$y_t = \beta_1 + \beta_2 x_t + e_t$$

we derive from the normal equations the formula

$$b_2 = \frac{T\sum x_t y_t - \sum x_t \sum y_t}{T\sum x_t^2 - (\sum x_t)^2} = \beta_2 + \frac{\sum (x_t - \bar{x}) e_t}{\sum (x_t - \bar{x})^2}$$

we use this result in our model

$$b_2 = \beta_2 + \frac{\sum (z_t - \bar{z}) e_t}{\sum (z_t - \bar{z})^2}$$

$$\text{plim } b_2 = \beta_2 + \frac{\text{plim } \frac{1}{T} \sum (z_t - \bar{z}) e}{\text{plim } \frac{1}{T} \sum (z_t - \bar{z})^2}$$

$$= \beta_2 - \frac{\beta_2 \sigma_u^2}{\sigma_{22}^2} \quad \Rightarrow \text{OLS INCONSISTENT}$$

1) because $\frac{1}{T}\sum (z_t - \bar{z})^2$ is the sample var of z and converges to the theoretical var of z, i.e. $E(z_t - E(z_t))^2 = var(z) = \sigma_{22}^2$

2) because $\frac{1}{T} \sum (z_t - \bar{z}) e_t$ converges to the covariance between z_t and e_t, i.e. $E\left[(z_t - E(z_t))(e_t - E(e_t))\right]$

all by the law of large numbers.

THE INSTRUMENTAL VARIABLE(S) (IV) ESTIMATOR

IDEA: REPLACE Z_t (the regressor) WITH A VARIABE SUCH THAT IT WOULD BE

- uncorrelated with the error term
- highly correlated with Z_t

so that:

$$\frac{1}{T}\sum \hat{e}_t = 0 \iff \frac{1}{T}\sum (Y_t - \hat{\beta}_1^{IV} - \hat{\beta}_2^{IV} Z_t) = 0$$

$$\frac{1}{T}\sum X_t \hat{e}_t = 0 \iff \frac{1}{T}\sum X_t (Y_t - \hat{\beta}_1^{IV} - \hat{\beta}_2^{IV} Z_t) = 0$$

where X_t is that variable and $\hat{\beta}_1^{IV}$, $\hat{\beta}_2^{IV}$ are the INSTRUMENTAL VARIABLE ESTIMATORS OF β_1 and β_2

rearrange:

$$T\hat{\beta}_1^{IV} + \sum Z_t \hat{\beta}_2^{IV} = \sum Y_t$$

$$\sum X_t \hat{\beta}_1^{IV} + \sum X_t Z_t \hat{\beta}_2^{IV} = \sum X_t Y_t$$

$$\rightarrow \quad \hat{\beta}_2^{IV} = \frac{\sum (X_t - \bar{X})(Y_t - \bar{Y})}{\sum (X_t - \bar{X})(Z_t - \bar{Z})}$$

$$\rightarrow \quad \hat{\beta}_1^{IV} = \bar{Y} - \hat{\beta}_2^{IV} \bar{Z}$$

Where $\hat{\beta}_1^{IV}$ and $\hat{\beta}_2^{IV}$ are IV estimators.

we check for consistency:

$$\hat{\beta}_2^{IV} = \beta_2 + \frac{\Sigma (x_t - \bar{x})\, e_t}{\Sigma (x_t - \bar{x})(z_t - \bar{z})}$$

$$\text{plim}\, \hat{\beta}_2^{IV} = \beta_2 + \frac{\text{plim}\, \frac{1}{T} \Sigma (x_t - \bar{x})\, e_t}{\text{plim}\, \frac{1}{T} \Sigma (x_t - \bar{x})(z_t - \bar{z})}$$

as
- $\frac{1}{T} \Sigma (x_t - \bar{x}) e_t \xrightarrow{T \to \infty} E\left[(x_t - E(x_t))(e_t - E(e_t))\right] = \text{cov}(x_t, e_t)$
- $\frac{1}{T} \Sigma (x_t - \bar{x})(z_t - \bar{z}) \xrightarrow{T \to \infty} E\left[(x_t - E(x_t))(z_t - E(z_t))\right] =$
$= \text{cov}(x_t, z_t)$

by the law of large numbers.

by assumption $\text{cov}(x_t, e_t) = 0$

we denote $\text{cov}(x_t, z_t)$ by σ_{xz}, so:

$$\text{plim}\, \hat{\beta}_2^{IV} = \beta_2 + \frac{0}{\sigma_{xz}} = \beta_2$$

IS CONSISTENT

The efficiency of IV depends on the choice of the instrument.
the MORE IS THE INSTRUMENT CORRELATED WITH THE REGRESSOR, THE LOWER IS THE VAR OF IV AND THE MORE EFFICIENT IT IS.

approximately, by the Central Limit Th.

$$\hat{\beta}_2^{IV} \overset{A}{\sim} N\left(\beta_2,\ var\left(\hat{\beta}_2^{IV}\right)\right)$$

where
$$\hat{var}\left(\hat{\beta}_2^{IV}\right) = \hat{\sigma}^2 \frac{\Sigma(x_t-\bar{x})^2}{\left[\Sigma(x_t-\bar{x})(z_t-\bar{z})\right]^2}$$

and $\hat{\sigma}^2 = \frac{1}{T}\Sigma \hat{e}_t^2$, $\hat{e}_t = y_t - \hat{\beta}_1^{IV} - \hat{\beta}_2^{IV} z_t$

the higher $cov(x_t, z_t)$ the larger the denominator $\rightarrow$
$\rightarrow \hat{var}\left(\hat{\beta}_2^{IV}\right)$ decreases

using matrix notation:

$$y = Z\beta + e$$

we chose X such that it is a $(T \times K)$ matrix

$$\text{plim}\ \frac{X'e}{T} = 0 \qquad \text{and} \qquad \text{plim}\ \frac{X'Z}{T} = \Sigma_{xz}$$

is finite and nonsingular

we denote $\text{plim}\ \frac{X'y}{T} = \Sigma_{xy}$ $\qquad \sigma \vdots \Sigma$

to <u>easily</u> derive IV premultiply $y = Z\beta + e$ by X'

$$X'y = X'Z\beta + X'e$$

$$X'Z\beta = X'y - X'e$$

$$\beta \cdot \text{plim}\ \frac{X'Z}{T} = \text{plim}\ \frac{X'y}{T} - \text{plim}\ \frac{X'e}{T}$$

$$\beta \cdot \Sigma_{xz} = \Sigma_{xy} - 0$$

$$\beta = (\Sigma_{xz})^{-1}\Sigma_{xy}$$

the IV estimator:

$$\hat{\beta}^{IV} = \left(\frac{X'Z}{T}\right)^{-1}\left(\frac{X'y}{T}\right) = (X'Z)^{-1}X'y$$

$$\hat{\beta}^{IV} = (X'Z)^{-1}X'Y = (X'Z)^{-1}X'(Z\beta + e)$$

$$= (X'Z)^{-1}X'Z\,\beta + (X'Z)^{-1}X'e$$

$$= \beta + (X'Z)^{-1}X'e$$

$$\operatorname{plim}\hat{\beta}^{IV} = \beta + \left(\operatorname{plim}\frac{X'Z}{T}\right)^{-1}\operatorname{plim}\frac{X'e}{T}$$

$$= \beta + \Sigma_{XZ}^{-1}\cdot 0$$

$$= \beta$$

variance of $\hat{\beta}^{IV}$:

$$= (X'Z)^{-1}X'ee'X(X'Z)^{-1}$$

$$= \left(\operatorname{plim}\frac{X'Z}{T}\right)^{-1}\left(\operatorname{plim}\frac{X'X}{T}\right)\left(\operatorname{plim}\frac{X'Z}{T}\right)^{-1}\cdot\operatorname{plim}\frac{ee'}{T}$$

$$= \Sigma_{XZ}^{-1}\,\Sigma_{XX}\,\Sigma_{XZ}^{-1}\cdot\sigma^2$$

estimated as

$$\widehat{\operatorname{cov}}(\hat{\beta}^{IV}) = \hat{\sigma}^2\cdot(X'Z)^{-1}X'X(Z'X)^{-1}$$

where $\hat{\sigma} = \frac{(y - Z\hat{\beta}^{IV})'(y - Z\hat{\beta}^{IV})}{T}$

HAUSMAN'S TEST

- a specification test
- test for presence of correlation btw a regressor and and error i.e. if model is well specified.

H_0: $\text{plim} \frac{1}{T} \sum (z_t - \bar{z}) e = 0$

H_A: $\text{plim} \frac{1}{T} \sum (z_t - \bar{z}) e \neq 0$

idea: is the OLS estimator b significantly different from β^{IV}.

(under the null b (OLS) is inconsistent)

$$m = [\hat{\beta}_{IV} - b]' [\hat{cov}(\hat{\beta}_{IV}) - \hat{cov}(b)]^{-1} [\hat{\beta}_{IV} - b]$$

under H_0, $m \sim \chi^2_{(J)}$

Reject H_0 when $m >$ critical value from $\chi^2_{(J)}$

IMPORTANT: FOR BOTH COV ESTIMATORS, USE SAME $\hat{\sigma}^2$

HOW DOES SAS ESTIMATE BY IV :

$$y_t = \beta_1 + \beta_2 z_t + e_t$$

$\mathrm{cov}(z_t, e_t) \neq 0$

consider instrument x_t such that 1) $\mathrm{cov}(x_t, e_t) = 0$

and 2) $\mathrm{cov}(x_t, z_t)$ IS "HIGH"

2- STEP PROCEDURE :

1) REGRESS z_t on x_t :

$$z_t = \alpha_1 + \alpha_2 x_t + u_t \quad , \text{where } X = [1, x_t]$$

get the fitted value of z_t, denoted by $\hat{z}_t$ in

$$\hat{z} = X[(X'X)^{-1} X' z] = X \hat{\alpha}_{OLS}$$

where $\hat{\alpha}_{OLS} = [\hat{\alpha}_{1_{OLS}} , \hat{\alpha}_{2_{OLS}}]$ and $\hat{Z} = [1, \hat{z}_t]$

2) REGRESS y_t on $\hat{z}_t$ INSTEAD OF REGRESSING y_t on z_t , i.e. run the regression :

$$y_t = \beta_1 + \beta_2 \hat{z}_t + e_t \quad , \text{where } \hat{Z} = [1, \hat{z}_t]$$

or $Y = \hat{Z} \beta + e$

the outcome is the estimator $\hat{\beta}$

$$\hat{\beta} = (\hat{Z}'\hat{Z})^{-1}\hat{Z}'Y$$

substituting $\hat{Z} = X(X'X)^{-1}X'Z$ into this formula:

$$\hat{\beta} = \left[Z'X(X'X)^{-1}\cancel{X'X}\,\cancel{(X'X)^{-1}}\,X'Z\right]^{-1} Z'X(X'X)^{-1}X'Y$$

$$\boxed{\hat{\beta} = \left[Z'X(X'X)^{-1}X'Z\right]^{-1} Z'X(X'X)^{-1}X'Y}$$

IF AND ONLY IF $Z'X$ are square matrices, as in this case because I have <u>one</u> variable to be instrumented and <u>one</u> instrument, I can write

$$\hat{\beta} = (Z'X)^{-1}\cancel{(X'X)}\,\cancel{(X'Z)^{-1}}\,\cancel{Z'X}\,\cancel{(X'X)^{-1}}\,X'Y$$

$$= (Z'X)^{-1}X'Y = \hat{\beta}^{IV}$$

compare to page 24

This 2-step estimator $\hat{\beta}$ is the "Two Stage Least Squares" (2SLS) estimator. If $Z'X$ is square, then 2SLS = IV estimator. If <u>not</u>, 2SLS has the formula above.