MULTIVARIATE TIME SERIES

Let us consider a bivariate vector of stationary t.s;

$$\begin{bmatrix} y_t \\ x_t \end{bmatrix}$$

- IT's mean is

$$E\begin{bmatrix} y_t \\ x_t \end{bmatrix} = \begin{bmatrix} E(y_t) \\ E(x_t) \end{bmatrix} = \begin{bmatrix} \mu_y \\ \mu_x \end{bmatrix}$$

- IT's variance is

$$\mathrm{Var}\begin{bmatrix} y_t \\ x_t \end{bmatrix} = \begin{bmatrix} \mathrm{var}(y_t) & \mathrm{cov}(y_t, x_t) \\ \mathrm{cov}(y_t, x_t) & \mathrm{var}(x_t) \end{bmatrix}$$

a symmetric, pos-def matrix.

$\mathrm{cov}(y_t x_t) = E[(y_t - \mu_y)(x_t - \mu_x)]$ is CALLED

CONTEMPORANEOUS COVARIANCE BTW x and y.

AUTOCOVARIANCE FUNCTION

consists of 2x2 matrices $\Gamma_0, \Gamma_1, \Gamma_2, \ldots \Gamma_K$

$$\Gamma_0 = \text{var}\begin{bmatrix} y_t \\ x_t \end{bmatrix}$$

$$\Gamma_1 = \begin{bmatrix} \gamma_y(1) & \gamma_{yx}(1) \\ \gamma_{xy}(1) & \gamma_x(1) \end{bmatrix}$$

where
$$\gamma_{yx}(1) = E\left[(y_t - \mu_y)(x_{t-1} - \mu_x)\right]$$
$$\gamma_{xy}(1) = E\left[(x_t - \mu_x)(y_{t-1} - \mu_y)\right]$$

$$\Gamma_K = \begin{bmatrix} \gamma_y(K) & \gamma_{yx}(K) \\ \gamma_{xy}(K) & \gamma_x(K) \end{bmatrix}$$

$\gamma_y(K)$ and $\gamma_x(K)$ are the autocovariances at lag k of y_t and x_t. $\gamma_{yx}(K)$ and $\gamma_{xy}(K)$ are cross-covariances at lag K.

AUTOCORRELATION FUNCTION

$$R_K = \begin{bmatrix} \dfrac{\gamma_y(K)}{\gamma_y(0)} & \dfrac{\gamma_{yx}(K)}{\sqrt{\gamma_y(0)\,\gamma_x(0)}} \\ \dfrac{\gamma_{xy}(K)}{\sqrt{\gamma_y(0)\,\gamma_x(0)}} & \dfrac{\gamma_x(K)}{\gamma_x(0)} \end{bmatrix} = \begin{bmatrix} \rho_y(K) & \rho_{yx}(K) \\ \rho_{xy}(K) & \rho_x(K) \end{bmatrix}$$

$$R_0 = \begin{bmatrix} 1 & \rho_{xy}(0) \\ \rho_{xy}(0) & 1 \end{bmatrix}$$

where $\rho_{xy}(0)$ and $\rho_{yx}(0)$ are equal and are the contemporaneous correlation between y_t and x_t

The autocovariances (autocorrelations) [except for lag 0] at lags 1, 2, etc are <u>not</u> symmetric however $\Gamma(K) = \Gamma'(-K)$ and $R(K) = R'(-K)$ and the functions are "even" in this sense.

1) y_t and x_t are both stationary

Var Model: (Vector Autoregressive) of order 1: VAR(1)

VAR(1):

$$y_t = \beta_{10} + \beta_{11} y_{t-1} + \beta_{12} x_{t-1} + v_t^y$$

$$x_t = \beta_{20} + \beta_{21} y_{t-1} + \beta_{22} x_{t-1} + v_t^x$$

where $\begin{bmatrix} v_t^y \\ v_t^x \end{bmatrix}$ is a bivariate white noise, i.e.

both series have mean 0 and variance Ω:

$$\begin{bmatrix} var(v_t^y) & cov(v_t^x, v_t^y) \\ cov(v_t^x, v_t^y) & var(v_t^x) \end{bmatrix} = var \begin{bmatrix} v_t^y \\ v_t^x \end{bmatrix} = \Omega$$

it is possible that $cov(v_t^x, v_t^y) = 0$. It is also possible to transform the noise into a noise with Identity variance $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, using the Cholesky decomposition. Then, the noises are said to be orthogonal

To see why it is called VAR(1):

$$\begin{bmatrix} y_t \\ x_t \end{bmatrix} = \begin{bmatrix} \beta_{10} \\ \beta_{20} \end{bmatrix} + \begin{bmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{bmatrix} \begin{bmatrix} y_{t-1} \\ x_{t-1} \end{bmatrix} + \begin{bmatrix} v_t^y \\ v_t^x \end{bmatrix}$$

where $\begin{bmatrix} v_t^y \\ v_t^x \end{bmatrix} \sim N\left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \Omega\right]$ is a bivariate White Noise

Denote $\begin{bmatrix} y_t \\ x_t \end{bmatrix}$ by Z_t :

$$Z_t = B_0 + \Phi Z_{t-1} + v_t$$

where Φ is a 2×2 autoregressive coefficient matrix. For stationarity, it has to have eigenvalues in modulus (absolute value) less than 1.

2) y_t and x_t are both non-stationary and have 1 unit root.

a) y_t and x_t NOT COINTEGRATED

VAR(1):

$$\Delta y_t = \beta_{10} + \beta_{11} \Delta y_{t-1} + \beta_{12} \Delta x_{t-1} + v_t^{\Delta y}$$

$$\Delta x_t = \beta_{20} + \beta_{21} \Delta y_{t-1} + \beta_{22} \Delta x_{t-1} + v_t^{\Delta x}$$

b) y_t and x_t ARE COINTEGRATED:

- LONG RUN EQUILIBRIUM = COINTEGRATING RELATIONSHIP (VECTOR)

in n series there can be AT MOST $n-1$ cointegrating (relationships)

$$y_t = \beta_0 + \beta_1 \cdot x_t + e_t$$

- Short-term adjustments concerns Δy_t and Δx_t as follows:

ECM :

$$\Delta y_t = \alpha_{10} + \alpha_{11} e_{t-1} + v_t^y$$

$$\Delta x_t = \alpha_{20} - \alpha_{21} e_{t-1} + v_t^x$$

IN PRACTICE, e_{t-1} can be replaced by $\hat{e}_{t-1}$ from the OLS regression of y_t on x_t

We say that the series react to the departure from the long run equilibrium at time $(t-1)$. that reaction takes place at time t.

α_{11} and α_{21} are the speed of adjustment coefficients : $\alpha_{11} \geq 0$, $\alpha_{21} \geq 0$. Either one of them has to be non-zero. Then only one series is closing the gap at time t.

ESTIMATION:

The VAR (ECM) can be estimated by separate OLS regressions, despite of the contemporaneous correlation of errors V_t^y and V_t^x

IFF the right-hand side **VARIABLES** are exactly the same in both regressions. !

IN general:

the Models are estimated by the Maximum Likelihood method **jointly**, i.e. both equations at a time.

$$L(B,\Omega) = -T \log 2\pi - \frac{T}{2} \log |\Omega| - \frac{1}{2} \sum_{t=2}^{T} \left\{ \begin{bmatrix} \Delta y_t \\ \Delta x_t \end{bmatrix} - \begin{bmatrix} \beta_{10} \\ \beta_{20} \end{bmatrix} - \begin{bmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{bmatrix} \begin{bmatrix} \Delta y_{t-1} \\ \Delta x_{t-1} \end{bmatrix} \right\}'$$

$$\Omega^{-1} \left\{ \begin{bmatrix} \Delta y_t \\ \Delta x_t \end{bmatrix} - \begin{bmatrix} \beta_{10} \\ \beta_{20} \end{bmatrix} - \begin{bmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{bmatrix} \begin{bmatrix} \Delta y_{t-1} \\ \Delta x_{t-1} \end{bmatrix} \right\}$$

Where $|\Omega|$ is the determinant of Ω.

The ECM model can be rewritten as a VEC model

example ECM(1) $\rightarrow$ VEC(1)

$$\begin{bmatrix} \Delta y_t \\ \Delta x_t \end{bmatrix} = \begin{bmatrix} \alpha_{10} - \alpha_{11}\beta_0 \\ \alpha_{20} + \alpha_{21}\beta_0 \end{bmatrix} + \begin{bmatrix} \alpha_{11} & -\alpha_{11}\beta_1 \\ -\alpha_{21} & \alpha_{21}\beta_1 \end{bmatrix} \begin{bmatrix} y_{t-1} \\ x_{t-1} \end{bmatrix} + \begin{bmatrix} v_t^y \\ v_t^x \end{bmatrix}$$

OR

$$\begin{bmatrix} \Delta y_t \\ \Delta x_t \end{bmatrix} = \begin{bmatrix} \Pi_{10} \\ \Pi_{20} \end{bmatrix} + \begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{21} & \Pi_{22} \end{bmatrix} \begin{bmatrix} y_{t-1} \\ x_{t-1} \end{bmatrix} + \begin{bmatrix} v_t^y \\ v_t^x \end{bmatrix}$$

the rank of matrix $\Pi = \begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{21} & \Pi_{22} \end{bmatrix}$ is <u>one</u>

Π is **not** of full rank as its rows are linear functions of one another.

The VEC model can be estimated ONLY by MLE

EXTENSIONS

- TO EXTEND THE VAR(1), ECM and VEC(1) to higher orders, we need to add y_{t-1} and x_{t-1} OR Δy_{t-1} and Δx_{t-1} to each row, respectively. This woud entail 4 more coefficients to be estimated from the model.

- to test for Granger "causality": Variabe x does not "cause" Y iff all the coefficients on the past and present x_t (Δx_t) are jointly zero, in the equation of Y.

(In the ECM you need to include the adjustment coeff too)

- VAR (p) and VEC(p):

  - IMPULSE RESPONSE FUNCTIONS

  - forecast error variance decomposition.