IDENTIFICATION AR(p)

To find the order p of an AR(p) we use the

PACF: partial autocorrelation function.

- The values of the PACF equal the last coefficient in a regression of Z_t on its lagged values:

$$Z_t = \delta + \underline{\phi_{11}} Z_{t-1} + a_t \Rightarrow \hat{\phi}_{11}$$

$$Z_t = \delta + \phi_{21} Z_{t-1} + \underline{\phi_{22}} Z_{t-2} + a_t \Rightarrow \hat{\phi}_{22}$$

$$Z_t = \delta + \phi_{31} Z_{t-1} + \phi_{32} Z_{t-2} + \underline{\phi_{33}} Z_{t-3} + a_t \Rightarrow \hat{\phi}_{33}$$

$$\vdots$$

- The argument of PACF is the lag:

	lag	PACF
(Z_{t-1})	1	ϕ_{11}
(Z_{t-2})	2	ϕ_{22}
(Z_{t-3})	3	ϕ_{33}
(Z_{t-4})	4	ϕ_{44}

Intuition: ϕ_{22} measures the linear dependence between Z_t and Z_{t-2} WHILE REMOVING THE EFFECT OF Z_{t-1}

PACF IS USED TO FIND THE ORDER p of AR(p)
PACF cuts off after p lags: the last value of PACF that is significantly different from 0 indicates the order p

Test of ϕ_{KK}:

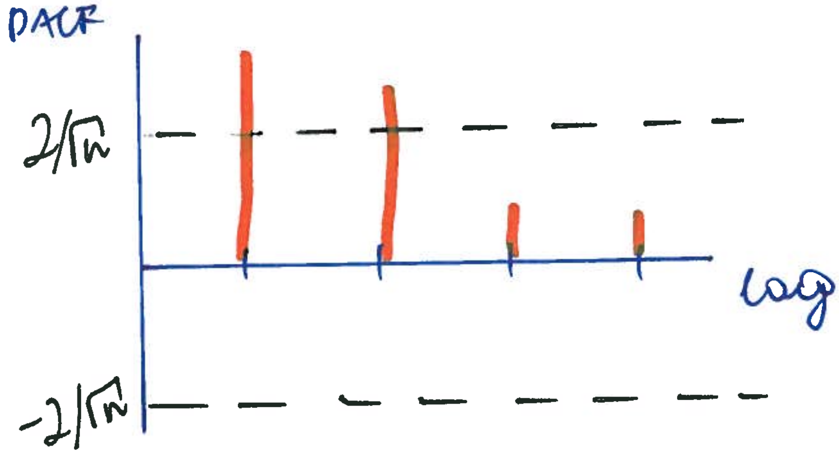
UNDER H_0: no linear dependence between Z_t and Z_{t-p}:

$$\hat{\phi}_{pp} \overset{a}{\sim} N\left(0, \frac{1}{n}\right)$$

$$\Rightarrow \quad \hat{\theta}_{pp} \pm 1.96 \cdot \frac{1}{\sqrt{n}}$$

IS THE CONFIDENCE INTERVAL

ex: AR(2):



ESTIMATION OF AR(p)

ASSUME THAT ORDER p IS KNOWN

$$z_t = \delta + \phi_1 z_{t-1} + \phi_2 z_{t-2} + \cdots \phi_p z_{t-p} + a_t$$

OLS

- the regressors are stochastic
- the arrays $\{z_t, z_{t-1}, z_{t-2}, \ldots z_{t-p}\}$ are not independent $\forall\, t$, but serially correlated instead
- error a_t is correlated with future values of regressors

OLS is biased in the presence of lagged endogenous variable among the regressors, but it is nevertheless CONSISTENT, i.e. valid in large sample.

$$y_{p+1} = \delta + \phi_1 y_p + \phi_2 y_{p-1} + \phi_3 y_{p-2} + \cdots + \phi_p y_1 + e_{p+1}$$

$$y_{p+2} = \delta + \phi_1 y_{p+1} + \phi_2 y_p + \phi_3 y_{p-1} + \cdots + \phi_p y_2 + e_{p+2}$$

$$\vdots$$

$$y_n = \delta + \phi_1 y_{n-1} + \phi_2 y_{n-2} + \phi_3 y_{n-3} + \cdots + \phi_p y_{n-p} + e_n$$

$$\begin{bmatrix} y_{p+1} \\ y_p \\ \vdots \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} 1 & y_p & y_{p-1} & \cdots & y_1 \\ 1 & y_{p-1} & y_p & & y_2 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & y_{n-1} & y_{n-2} & \cdots & y_{n-p} \end{bmatrix} \begin{bmatrix} \delta \\ \phi_1 \\ \phi_2 \\ \vdots \\ \phi_p \end{bmatrix} + \begin{bmatrix} e_{p+1} \\ e_{p+2} \\ \vdots \\ \vdots \\ e_n \end{bmatrix}$$

$$Y = X\beta + e$$

$$\hat{\beta} = (X'X)^{-1}X'Y$$

$$\hat{var}(\hat{\beta}) = \hat{\sigma}_e^2 (X'X)^{-1}$$

$$\hat{\sigma}_e^2 = \frac{\hat{e}'\hat{e}}{n-2p-1} = \frac{(Y-X\hat{\beta})'(Y-X\hat{\beta})}{n-2p-1}$$

because $T-p$ can be used for estimation of $AR(p)$

$p+1$ coefficients to estimate (ϕ's + intercept)

$$\hat{\mu} = \frac{\hat{\delta}}{1-\hat{\phi}_1 - \cdots - \hat{\phi}_p}$$

MLE

consider:

$$a_{p+1} = Z_{p+1} - \delta - \phi_1 Z_p - \phi_2 Z_{p-1} - \cdots - \phi_p Z_1$$

$$a_{p+2} = Z_{p+2} - \delta - \phi_1 Z_{p+1} - \phi_2 Z_p - \cdots - \phi_p Z_2$$

$$\vdots$$

$$a_n = Z_n - \delta - \phi_1 Z_{n-1} - \phi_2 Z_{n-2} - \cdots - \phi_p Z_{n-p}$$

ASSUME THAT $a_1, a_2, \ldots, a_n$ are IID $N(0, \sigma_a^2)$

i.e. are <u>STRONG</u> White Noise and are normally distribute

$$f(a_1, a_2, \ldots a_n) = f(a_1) \cdot f(a_2) \cdots f(a_n)$$

$$\ln L(\delta, \phi_1, \phi_2, \ldots \phi_p \mid Z_1, Z_2 \ldots Z_n) =$$

$$-\frac{n}{2}\ln(2\pi) - \frac{n}{2}\ln\sigma_a^2 - \frac{1}{2}\sum_{t=p+1}^{n}\frac{(Z_t - \delta - \phi_1 Z_{t-1} - \cdots - \phi_p Z_{t-p})^2}{\sigma_a^2}$$

- backcasting to recover $a_1 \ldots a_p$ can be used.
- estimators are CONSISTENT
  - asymptotically efficient
  - asymptotically normally distributed.

$$\sqrt{n}\,(\hat{\phi}_n - \phi_0) \overset{a}{\sim} N(0, I^{-1})$$

where $\hat{\phi}_n = \begin{bmatrix} \hat{\phi}_1 \\ \hat{\phi}_2 \\ \vdots \\ \hat{\phi}_p \\ \hat{\delta} \end{bmatrix}$, $\phi_0 = \begin{bmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \delta \end{bmatrix}$ and $I = E\left[-\frac{\partial^2 \ln L}{\partial\phi\,\partial\phi'}\right]$

is the information matrix.

test are asymptotically valid.

Wald $\quad \dfrac{\sqrt{n}\,(\tilde{\phi}_K - \phi_{0K})}{\sqrt{\operatorname{var} \hat{\phi}_K}} \overset{a}{\sim} N(0,1)$

to test H_0: $\phi_K = 0$ use the "t-ratio"

$$\text{"}t\text{"} = \frac{\hat{\phi}_K}{\sqrt{\widehat{\operatorname{var}}\, \hat{\phi}_K}}$$

under H_0 "t" is asymptotically $N(0,1)$ distributed

In practice, the Information matrix inverse is approximated by the inverse of the Hessian matrix or the outer product of scores:

$$\hat{I}_n = -\frac{1}{n} \sum_{t=1}^{n} \frac{\partial^2 \ln L}{\partial \phi \, \partial \phi'} \simeq \frac{1}{n} \sum_{t=1}^{n} \frac{\partial \ln L}{\partial \phi} \frac{\partial \ln L'}{\partial \phi}$$

the variances of each estimated ϕ_K are on the main diagonal of $\hat{I}_n^{-1}$

note that this MLE and OLS are asymptotically equivalent, because $\min \sum_{t=p+1}^{n} e_t^2 = \sum_{t=p+1}^{n} (z_t - \delta - \phi_1 z_{t-1} - \cdots - \phi_p z_{t-p})$

is equivalent to maximizing $\ln L$ with respect to $\phi_1 \ldots \phi_p$ and δ

Next, the estimator of σ_a^2 can be obtained.

other useful tests:

likelihood ratio:

$$2\left[\ln L(\hat{\phi}_n) - \ln L(\hat{\phi}_n^c)\right] \overset{a}{\sim} \chi^2(r)$$

Note also that at the maximum, we have

$$\ln L = -\frac{n}{2}\ln(2\pi) - \frac{n}{2}\ln\hat{\sigma}_a^2 - \frac{1}{2}\frac{n\cancel{\hat{\sigma}_a^2}}{\cancel{\hat{\sigma}_a^2}}$$

$$\simeq -\frac{n}{2}\ln\hat{\sigma}_a^2 - \frac{n}{2}$$

MA(q)

Moving average of order q

$$Z_t = \mu + a_t - \theta_1 a_{t-1} - \theta_2 a_{t-2} \cdots - \theta_q a_{t-q}$$

- mean

$$E(Z_t) = \mu$$

- variance

$$\begin{aligned}
\gamma_0 = var(Z_t) &= E(Z_t - \mu)^2 \\
&= E\Big(a_t^2 + \theta_1^2 a_{t-1}^2 + \cdots + \theta_q^2 a_{t-q}^2 \\
&\qquad \underbrace{-\,2\theta_1\theta_2 a_{t-1} a_{t-2}}_{=0} + \underbrace{\cdots}_{=0}\Big) \\
&= \sigma_a^2 + \theta_1^2 \sigma_a^2 + \cdots + \theta_q^2 \sigma_a^2 \\
&= \sigma_a^2 (1 + \theta_1^2 + \cdots + \theta_q^2)
\end{aligned}$$

ex: MA(1)

$$Z_t = \mu + a_t - \theta_1 a_{t-1}$$

- $E(Z_t) = \mu$
- $\gamma_0 = var(Z_t) = E(Z_t - \mu)^2 = \sigma_a^2(1 + \theta_1^2)$

$$\gamma_1 = cov(Z_t, Z_{t-1}) = E\left[(Z_t-\mu)(Z_{t-1}-\mu)\right]$$

$$= E\left[(a_t - \theta_1 a_{t-1})(a_{t-1} - \theta_1 a_{t-2})\right]$$

$$= \theta_1^2 \sigma_a^2$$

$$\gamma_2 = cov(Z_t\, Z_{t-2}) = E\left[(Z_t-\mu)(Z_{t-2}-\mu)\right]$$

$$= E\left[(a_t - \theta_1 a_{t-1})(a_{t-2} - \theta_1 a_{t-3})\right]$$

$$= 0$$

THE ACF OF A MA(q) CUTS OFF AT LAG q INDICATING THE ORDER OF MA(q)

$\Rightarrow$ USE ACF TO FIND THE ORDER OF MA (IDENTIFICATION)

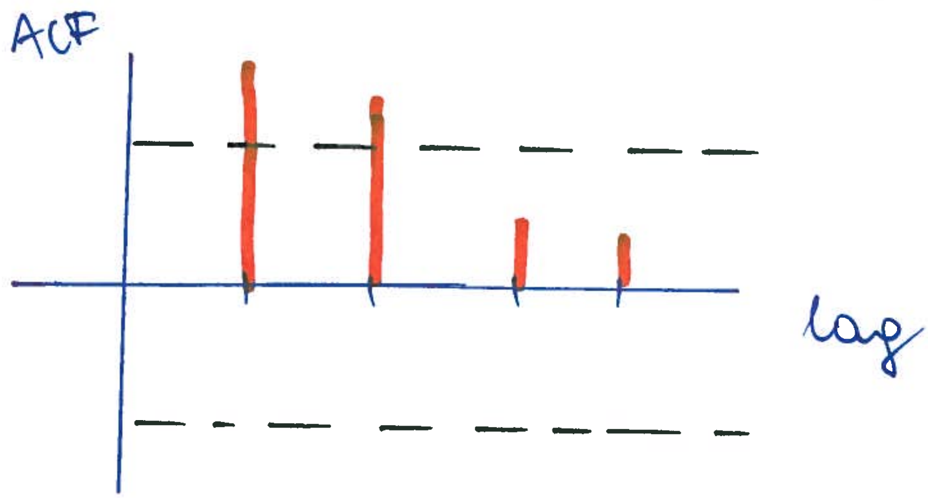
$$\rho_K = \frac{\gamma_K}{\gamma_0} = \begin{cases} \dfrac{\theta_1}{1+\theta_1^2}, & K=1 \\ 0, & K>1 \end{cases}$$

Recall, under $H_0 : \rho_K = 0$ we have

$$\hat{\rho}_K \overset{a}{\sim} N\left(0, \frac{1}{n}\right)$$

and $\hat{\rho}_K \pm 1.96 \cdot \frac{1}{\sqrt{n}}$ is the confidence interval.

MA(2):



MA(2):

$$Z_t = \mu + a_t - \theta_1 a_{t-1} - \theta_2 a_{t-2}$$

- $E(Z_t) = \mu$
- $\gamma_0 = var(Z_t) = \sigma_a^2 (1 + \theta_1^2 + \theta_2^2)$
- $\gamma_1 = cov(\dot{Z}_t, \dot{Z}_{t-1}) = E[(a_t - \theta_1 a_{t-1} - \theta_2 a_{t-2})(a_{t-1} - \theta_1 a_{t-2} - \theta_3 a_{t-3})]$

  $= \theta_1 \sigma_a^2 + \theta_1 \theta_2 \sigma_a^2 = \sigma_a^2 (\theta_1 + \theta_1 \theta_2)$
- $\gamma_2 = cov(\dot{Z}_t, \dot{Z}_{t-2}) = E[(a_t - \theta_1 a_{t-1} - \theta_2 a_{t-2})(a_{t-2} - \theta_1 a_{t-3} - \theta_2 a_{t-4})] = -\theta_2 \sigma_a^2$

$$\gamma_3 = cov(\dot{z}_t, \dot{z}_{t-3}) = E\left[(a_t - \theta_1 a_{t-1} - \theta_2 a_{t-2})(a_{t-3} - \theta_1 a_{t-4} - \theta_1 a_{t-5})\right]$$

$$= 0$$

for MA(2) :

$$\rho_1 = \frac{\theta_1(1+\theta_2)}{1+\theta_1^2+\theta_2^2}$$

$$\rho_2 = \frac{-\theta_2}{1+\theta_1^2+\theta_2^2}$$

$$\rho_K = 0 \quad \text{for } K > 2$$

ANY MA(q) IS ALWAYS STATIONARY

FOR ANY MA(q):

$$\rho_K = \begin{cases} \dfrac{\sum_{i=0}^{q-K} \theta_i \theta_{i+K}}{\sum_{i=0}^{q} \theta_i^2}, & |K| \leq q \\ 0 & |K| > q \end{cases}$$