GCov-Based Portmanteau Test

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Abstract

We study nonlinear serial dependence tests for non-Gaussian time series and residuals of dynamic models based on portmanteau statistics involving nonlinear autocovariances. A new test with an asymptotic χ^2 distribution is introduced for testing nonlinear serial dependence (NLSD) in time series. This test is inspired by the Generalized Covariance (GCov) residual-based specification test, recently proposed as a diagnostic tool for semi-parametric dynamic models with i.i.d. non-Gaussian errors. It has a χ^2 distribution when the model is correctly specified and estimated by the GCov estimator. We derive new asymptotic results under local alternatives for testing hypotheses on the parameters of a semi-parametric model. We extend it by introducing a GCov bootstrap test for residual diagnostics,which is also available for models estimated by a different method, such as the maximum likelihood estimator under a parametric assumption on the error distribution. A simulation study shows that the tests perform well in applications to mixed causal-noncausal autoregressive models. The GCov specification test is used to assess the fit of a mixed causal-noncausal model of aluminum prices with locally explosive patterns, i.e. bubbles and spikes between 2005 and 2024.

Keywords: Semi-Parametric Estimator, Generalized Covariance Estimator, Portmanteau Statistic, Causal-Noncausal Process, Bubbles.

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1 Introduction

Diagnostic checking is an important component of the traditional Box-Jenkins procedure for the identification and estimation of stationary linear ARMA models with serially uncorrelated Gaussian errors. It has been routinely conducted by applying the Box-Pierce or Liung-Box tests to the residuals of an ARMA model to detect serial correlation. Analogously, the diagnostic checking of nonlinear ARCH-type models is often conducted by applying the Box-Pierce or Liung-Box test twice to the residuals and their squares. Recently, there has been a growing interest in dynamic models with independent, identically distributed (i.i.d.) non-Gaussian errors, such as the structural vector autoregressive (SVAR) models for macroeconomic data, the univariate mixed causal-noncausal autoregressive (MAR), multivariate mixed vector autoregressive (VAR) and double autoregressive (DAR) models for processes with local explosive features, such as spikes and bubbles, displayed in practice by the commodity and cryptocurrency price processes, for example. Diagnostic checking in these models is commonly based on the Liung Box tests applied to the residuals, their squares, and higher powers, which is a cumbersome multi-step procedure.

This paper reviews convenient one-step diagnostic procedures for jointly testing various forms of nonlinear serial dependence in non-Gaussian time series before model estimation, as well as in the residuals ex-post, as a specification test for dynamic models with i.i.d. non-Gaussian errors.

We introduce a new (non)linear serial dependence (NLSD) test for strictly stationary time series with non-Gaussian distributions. We show that in strictly stationary processes, the hypothesis of the absence of linear and nonlinear serial dependence can be tested by a multivariate portmanteau test involving nonlinear autocovariances, i.e., the autocovariance (matrices) of nonlinear transformations of a time series. This approach is available for either univariate or multivariate processes, and the test statistic follows asymptotically a chi-square distribution under the null hypothesis.

This method is inspired by the Generalized Covariance (GCov) estimator and a specification test introduced by Gourieroux and Jasiak (2023). The GCov specification test allows for detecting nonlinear serial correlation in the residuals and is a convenient one-step procedure that can replace multiple Box-Pierce and Liung-Box tests for diagnostic checking of strictly stationary semi-parametric models with i.i.d. non-Gaussian errors. The advantage of the semi-parametric approach is that it does not require any distributional assumptions on the errors other than being i.i.d. and non-Gaussian. Like the NLSD test, the GCov test statistic has an asymptotic chi-square distribution under the null hypothesis, assuming that the dynamic model is estimated by the Generalized Covariance (GCov) estimator [Gourieroux, Jasiak (2023)]. The GCov is a semi-parametric one-step estimator, which is consistent, asymptotically normally distributed and semi-parametrically efficient[3]. We study analytically and through simulations the finite sample properties of this test under the local alternative hypotheses to provide convincing arguments and empirical evidence revealing its potential as a widely applicable diagnostic tool.

The limitation of the GCov-specification test is that its asymptotic distribution is known only when the dynamic model is estimated by the Generalized Covariance (GCov) estimator. It is only then that the multivariate portmanteau test statistic computed from the nonlinear autocovariance (matrices) of residuals follows asymptotically a χ^2 distribution under the null hypothesis. This motivates us to introduce a new bootstrap-based GCov test that allows for using the GCov specification test in models with i.i.d. errors estimated consistently by a different method, such as the generalized Method of Moments (GMM) or the maximum likelihood (ML), approximate (AML) or quasi maximum likelihood (QML or PML) methods under the parametric assumptions on the error distribution.

This paper contributes to the literature on univariate and multivariate portmanteau tests. The GCov specification test can be compared to the test of the martingale difference hypothesis of De Gooijer (2023), which relies on a portmanteau test statistic computed from the residuals and squared residuals. The GCov is more general in the sense that it can include the autocovariances of various nonlinear functions of the residuals rather than the residuals and their squares only. An alternative approach for specification testing is based on the distance covariance. Davis and Wan (2022) consider the auto-distance covariance function and propose a specification test of the null hypothesis of residual independence. Compared to that approach, the GCov specification test has an advantage in that its asymptotic distribution is known, while the asymptotic distribution of the Davis and Wan test statistic needs to be found by bootstrap. Chu (2023) also uses the distance covariance approach and considers the null hypothesis of residual independence. Although the theoretical test statistic proposed by Chu has a known limiting distribution, in practice, it needs to be approximated, and the critical values have to be found by bootstrap. For data analysis in the frequency domain, the inference methods of Velasco and Lobato (2018) are available, but are not discussed here as the focus of our paper is on the time domain only.

This paper also makes a contribution to the practice of time series analysis. There exists a growing literature on the SVAR models, motivated by the fact that the identification problem

[3] It achieves the parametric efficiency for well-chosen nonlinear transformations.

arising in these models can be solved by assuming that the structural shocks are independent and non-Gaussian. In applied research, residual diagnostics are often disregarded, likely because applying multiple Liung Box tests to the data and then to the model residuals and their powers, for example, is too cumbersome. Then, the model selection is based only on the goodness of fit criteria. For example, Guay (2021) and Keweloh (2020) assume in their applications that the residuals are independent and chi-square distributed without testing them for the absence of (non)linear serial dependence, or else the test results are not reported. Gourieroux, Monfort and Renne (2017) use unspecified portmanteau tests to justify the fit of their model without reporting the residual diagnostic check results either. Gourieroux, Monfort and Renne (2018) write that the Box and Pierce (1970) and Ljung and Box (1978) tests of the null hypothesis of no auto-correlation in the residuals were applied. Lanne, M., and J. Luoto (2019) comment on the absence of residual serial correlation without specifying how it was determined. Lanne, Meitz, and Saikkonen (2017) provide the most detailed description of applying the Ljung-Box test to the residuals and provide the p-values. This applied research would benefit from a convenient one-step test designed for detecting nonlinear dependence in the residuals.

While the GCov-based specification test is applicable to a variety of models with i.i.d. non-Gaussian errors, it is particularly useful for testing the fit of causal-noncausal dynamic models with non-Gaussian errors. This is because the nonlinear autocovariances identify the noncausal dynamics [Chan et al. (2006)]. There is a growing interest in the univariate and multivariate mixed causal-noncausal models for processes with locally explosive patterns, such as the short-lasting trends, bubbles and spikes, and time-varying volatility [see e.g. Hecq et al. (2016), (2020), Gourieroux, Jasiak (2017), (2023), Gourieroux, Zakoian (2017), Fries, Zakoian (2019), Hecq and Voisin (2021), Swensen (2022), Davis, Song (2020)]. In practice, the locally explosive patterns characterize the time series of commodity prices, including oil, soybean, nickel, and aluminum prices and cryptocurrency prices of native cryptocurrency, tokens, and some stablecoins. The estimators available for this class of models are the aforementioned semi-parametric GCov estimator [Gourieroux, Jasiak (2017), (2023)] and the parametric (Approximate) Maximum Likelihood (AML), or ML estimators [Lanne, Saikkonen (2011), Davis, Song (2020)]. When the error distribution of a mixed model is misspecified, the AML estimator can be unreliable [Hecq, Lieb and Telg (2016)], adversely affecting the outcomes of any ML-based fit criteria, such as the AIC, Schwartz and Hannan-Quinn criteria. Hence, in these models, the proposed tests arise as convenient tools for both detecting nonlinear serial dependence in the data and testing the goodness of fit.

We examine the finite sample performance of the GCov specification test analytically under a sequence of local alternatives to study the hypotheses about the parameters of a semi-parametric model. Our simulation study shows that the NLSD, GCov specification, and GCov bootstrap tests perform well in detecting nonlinear serial dependence in mixed causal-noncausal processes and the residuals of univariate (MAR) and multivariate autoregressive VAR models, respectively. The semi-parametric GCov specification test for the residuals successfully detects the correct fit in various models and is a valuable diagnostic tool. We also evidence the good performance of the GCov bootstrap test applied to the residuals of a model estimated by the AML estimator. We illustrate empirically the GCov specification test applied to a mixed causal-noncausal autoregressive (MAR) model of commodity prices estimated by the AML method.

The following notation is used. For any $m \times n$ matrix A with the jth column $a_j, j = 1, ..., n$ $vec(A)$ will denote the column vector of dimension mn defined as:

$$vec(A) = (a_1' ..., a_j', ..., a_n')',$$

where the prime denotes a transpose. For any two matrices $A \equiv (a_{ij})$ and B, the Kronecker product $(A \otimes B)$ is the block matrix with the (i, j)th block denoted by $a_{ij}B$.

The paper is organized as follows. Section 2 introduces the new linear and non-linear serial dependence NLSD test for time series inspired by the classical multivariate portmanteau test, which is briefly reviewed. Section 3 reviews the GCov estimator and the existing GCov specification test based on the portmanteau test statistic with nonlinear autocovariances of residuals of a model estimated by the GCov estimator. It provides new results on the asymptotic properties of this test under a sequence of local alternatives. Section 4 examines how the asymptotic performance of the test can be improved by increasing the number of nonlinear autocovariances. Section 5 introduces the new GCov bootstrap-adjusted specification test. Section 6 presents the simulation results. Section 7 shows an empirical application of the specification test to a mixed causal-noncausal model fitted to monthly aluminum prices recorded between 2005 and 2024. Section 8 concludes. The technical results are given in Appendices A, B, and additional empirical results appear in Appendix C.

2 NLSD Test for Non-Gaussian Processes

This section introduces the new linear and nonlinear serial dependence (NLSD) test for non-Gaussian processes based on a portmanteau statistic involving nonlinear transformations of strictly stationary univariate or multivariate time series with non-Gaussian marginal distributions and nonlinear dynamics.

The linear and nonlinear dependence test considered in this paper technically concerns the null hypothesis of zero values of autocovariances/autocorrelations of the transformed series. For processes with Gaussian distributions, zero-valued autocovariances are equivalent to serial independence, which becomes the null hypothesis of interest. Then, the asymptotic distribution of the test statistics for testing the independence hypothesis is determined under this null hypothesis. In the case of non-Gaussian processes, zero-valued linear autocovariances do not imply serial independence. For this reason, we consider testing for the absence of (non)linear serial dependence. By analogy to the traditional literature, we use the independence hypothesis to derive the asymptotic distributions of the test statistics.

2.1 Linear Serial Dependence Test for Univariate and Multivariate Time Series

Let us recall the results that exist in the literature on the test of weak white noise hypothesis, i.e. of the absence of linear dependence.

2.1.1 Univariate Time Series

Let us consider a univariate stationary time series (y_t) with finite fourth-order moments[4]. The test of weak white noise hypothesis $H_0 = \{\gamma(h) = 0, h = 1, ..., H\}$, with $\gamma(h) = Cov(y_t, y_{t-h})$ is commonly based on the test statistic:

$$\hat{\xi}_T(H) = T \sum_{h=1}^{H} \hat{\rho}(h)^2 = T \sum_{h=1}^{H} \frac{\hat{\gamma}(h)^2}{\hat{\gamma}(0)^2}, \tag{2.1}$$

where $\hat{\gamma}(h)$ and $\hat{\rho}(h)$ are the sample autocovariance and autocorrelation of order h, respectively, computed from a sample of T observations[5].

Under the null hypothesis of independence and standard regularity conditions, this statistic follows asymptotically a chi-square distribution $\chi^2(H)$ with H degrees of freedom [see Box, Pierce, 1970]. The following two subsections introduce an analogous test statistic for testing for the absence of (non)linear serial dependence in strictly stationary univariate or multivariate processes.

[4]The existence of moments up to order 4 is needed for deriving the asymptotic variance of $\hat{\gamma}(h)$ under the asymptotic normality.

[5]In this Section the index T of the estimators, e.g. $\hat{\gamma}_T(h)$ is omitted to simplify the notation.

2.1.2 Multivariate Time Series

Let us now consider a strictly stationary time series (Y_t) of dimension n with finite fourth-order moments. The null hypothesis is $H_0 = \{\Gamma(h) = 0, h = 1, ..., H\}$, where $\Gamma(h) = Cov(Y_t, Y_{t-h})$ is the autocovariance of order h. The multivariate equivalent of the test statistic (2.1) is:

$$\hat{\xi}_T(H) = T \sum_{h=1}^{H} Tr[\hat{R}^2(h)], \tag{2.2}$$

where $\hat{R}^2(h)$ is the sample analogue of the multivariate R-square defined by:

$$R^2(h) = \Gamma(h)\Gamma(0)^{-1}\Gamma(h)'\Gamma(0)^{-1}. \tag{2.3}$$

Since

$$\hat{R}^2(h) = \hat{\Gamma}(0)^{1/2}[\hat{\Gamma}(0)^{-1/2}\hat{\Gamma}(h)\hat{\Gamma}(0)^{-1}\hat{\Gamma}(h)'\hat{\Gamma}(0)^{-1/2}]\hat{\Gamma}(0)^{-1/2}, \tag{2.4}$$

this matrix is equivalent up to a change of basis to the matrix within the brackets, which is symmetric and positive-definite. Therefore, it is diagonalizable, with a trace equal to the sum of its eigenvalues, which are the squares of the canonical correlations between Y_t and Y_{t-h}, denoted by $\hat{\rho}_i^2(h), i = 1, ..., n$ [Hotelling (1936)]. Therefore:

$$\hat{\xi}_T(H) = T \sum_{h=1}^{H} Tr[\hat{\Gamma}(h)\hat{\Gamma}(0)^{-1}\hat{\Gamma}(h)'\hat{\Gamma}(0)^{-1}] = T \sum_{h=1}^{H} [\sum_{i=1}^{n} \hat{\rho}_i(h)^2].$$

Under the null hypothesis of strong white noise, this statistic follows asymptotically a chi-square distribution $\chi^2(nH)$ [see, e.g. Robinson (1973), Anderson (1999), Section 7, Anderson (2002), Section 5].

2.2 Nonlinear Serial Dependence Test for Time Series

Let us extend the results presented so far and introduce a portmanteau (NLSD) test based on nonlinear functions of a strictly stationary time series y_t with a non-Gaussian distribution. The results of Section 2.1 suggest that the null hypothesis of the absence of (non)linear dependence in univariate or multivariate time series y_t can be tested by applying the test statistic $\hat{\xi}_T(H)$ to a vector of nonlinear transformations of y_t. It follows from Chan et al. (2006) and Gourieroux, Jasiak (2023) that the nonlinear autocovariances, i.e. the autocovariances of nonlinear functions of (y_t) identify and thus account for the nonlinear and noncausal (i.e., future dependent) serial dependence as well.

To build the NLSD test, we compute the test statistics $\hat{\xi}_T(H)_a$ from nonlinear transforms Y_t^a of a univariate or multivariate non-Gaussian time series y_t, where a is a nonlinear function satisfying the regularity conditions given in Gourieroux, Jasiak (2023), and are continuous and differentiable.

Let us consider a vector of such nonlinear functions a of a univariate strictly stationary process y_t. That vector increases the dimension of y_t by appending it with the nonlinear functions of y_t, such as the squares or logarithms. In particular if (y_t) has no finite fourth-order moment, then it can be replaced by a transformed multivariate process Y_t^a with a finite fourth-order moment to ensure the validity of the asymptotic distributional results under the null hypothesis.

For ease of exposition, let us introduce K nonlinear functions $a_1, ..., a_K$ of the process (y_t), transforming it into a multivariate process of a higher dimension, denoted by K with components $a_k(y_t)$:

$$Y_t^a = \begin{pmatrix} a_1(y_t) \\ \vdots \\ a_K(y_t) \end{pmatrix},$$

where $a_1(y_t) = y_t$ is the time series itself, allowing the test to capture the linear dependence. We compute the sample autocovariances of the transformations $a_k(y_t), k = 1, ..., K$:

$$\hat{\Gamma}^a(h) = \frac{1}{T}\sum_{t=h}^{T} Y_t^a Y_{t-h}^{a'} - \frac{1}{T}\sum_{t=h}^{T-1} Y_t^{a'} \frac{1}{T}\sum_{t=h+1}^{T} Y_{t-h}^a.$$

Once a set of transformations is determined, the null hypothesis becomes:

$$H_{0,a} = (\Gamma^a(h) = 0,\ h = 1, ..., H),$$

allows us to test for the absence of (non)linear dependence. Because one cannot consider all the lags and nonlinear transformations, the null hypothesis of the absence of (non)linear dependence is not equivalent to the independence condition, but is arbitrarily close to it, depending on lag H and the number of nonlinear transformations considered. Then, the NLSD test statistic:

$$\hat{\xi}_T(H)_a = T\sum_{h=1}^{H} Tr\hat{R}_a^2(h) \tag{2.5}$$

where

$$\hat{R}_a^2(h) = \hat{\Gamma}_a(h)\hat{\Gamma}_a(0)^{-1}\hat{\Gamma}_a(h)'\hat{\Gamma}_a(0)^{-1}],$$

is computed from the nonlinear sample autocovariances, i.e. the autocovariance matrices of a transformed univariate or multivariate process. In each case, the dimension of the process is increased, i.e. becomes higher than that of the initial time series of interest. If the combined dimension of the process is K, then under serial independence, the NLSD test statistic (2.5) follows asymptotically a $\chi^2(K^2H)$ distribution [see, e.g. Robinson (1973), Chitturi (1976), Anderson (1999), Section 7, Anderson (2002), Section 5]. The test of the null hypothesis H_0 at level α is performed as follows: the null hypothesis H_0 is rejected when $\hat{\xi}_T(H) > \chi^2_{1-\alpha}(K^2H)$ and H_0 is not rejected otherwise.

For practical implementations, it is easy to show that the NLSD test statistic is invariant with respect to the scale effect and change of sign of y_t^a. Let us consider a diagonal matrix A:

$$A = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^2 \end{bmatrix}$$

where λ represents the scale effect or the change of sign effect for $\lambda = -1$. Then the multivariate R-square of process $y_t^a = [y_t, y_t^2]'$:

$$R_a^2(1) = \Gamma_a(1)\Gamma_a(0)^{-1}\Gamma_a(1)'\Gamma_a(0)^{-1}$$

computed for the rescaled process Ay_t^a is:

$$\begin{aligned}
\tilde{R}_a^2(1) &= A\,\Gamma_a(1)\,A[A\,\Gamma_a(0)\,A]^{-1}\,A'\Gamma_a(1)'\,A'\,[A\,\Gamma_a(0)\,A]^{-1} \\
&= A\Gamma_a(1)AA^{-1}\Gamma_a(0)^{-1}A^{-1}A\Gamma_a(1)'AA^{-1}\Gamma_a(0)^{-1}A^{-1} \\
&= A\Gamma_a(1)\Gamma_a(0)^{-1}\Gamma_a(1)'\Gamma_a(0)^{-1}A^{-1}
\end{aligned}$$

Hence, we find that the multivariate R-square of the transformed process is $AR_a^2(1)A^{-1}$. We see that its trace is

$$Tr(\tilde{R}_a^2(1)) = Tr(AR_a^2(1)A^{-1}) = Tr(R_a^2(1)AA^{-1}) = TrR_a^2(1),$$

which implies that the test statistic $TTr\tilde{R}_a^2(1) = TTr(AR_a^2(1)A^{-1})$ remains unchanged and is equal for y_t^a and Ay_t^a .

3 GCov Specification Test for Semi-Parametric Models

The NLSD test introduced in Section 2.2 is a special case of the goodness of fit Generalized Covariance (GCov) specification test introduced by Gourieroux, Jasiak (2023) for semi-parametric nonlinear models of strictly stationary time series with i.i.d. errors and parameter vector θ describing their dynamics. In this context, $\hat{\xi}_T(H)$ is computed from a multivariate time series of residuals and their nonlinear transforms instead of an observed time series, which changes its limiting distribution to $\chi^2(K^2H - dim(\theta))$ [Gourieroux, Jasiak (2023)]. The model and GCov test are reviewed below.

3.1 The Semi-Parametric Model

Let us consider a strictly stationary process (Y_t) satisfying a semi-parametric model studied in Gourieroux, Jasiak (2023):

$$g(\tilde{Y}_t; \theta) = u_t, \tag{3.1}$$

where g is a known function satisfying the regularity conditions, where $dim(g) = dim(u_t) = J$, $\tilde{Y}_t = (Y_t, Y_{t-1}, \ldots, Y_{t-p})$, p is a non-negative integer, (u_t) is an i.i.d. sequence (not necessarily with mean zero) with a common marginal density function f and θ is an unknown parameter vector of dimension $dim(\theta)$. We assume that the model is well-specified, the true value of parameter θ is θ_0 and the true error density is f_0. Moreover, u_t is not assumed to be independent of $\tilde{Y}_{t-1}$. Hence, errors u_t are not necessarily interpretable as either causal, or non-causal innovations.

Similarly, as it was done for testing for the absence of nonlinear dependence in Section 2.2, Model (3.1) can be transformed into a system of higher dimension by considering (linear and) nonlinear scalar transformations of u_t. Let us introduce K nonlinear transformations $a_1, ..., a_K$. Then we have:

$$\begin{aligned} a_k[g(\tilde{y}_t; \theta)] &= a_k(u_t),\ k = 1, ..., K, \\ \text{or, equivalently}\ \ a[g(\tilde{y}_t; \theta)] &= a(u_t) \equiv v_t, \end{aligned} \tag{3.2}$$

where the transformed process (v_t) is also an i.i.d. process. Henceforth, the subscripts of transformations a are omitted for clarity.

3.2 The GCov Test

The GCov test of model specification is a test of the null hypothesis of the absence of (non)linear serial dependence in the ((non)linear transformations of) the residuals of a semi-parametric model with i.i.d. errors whose distribution is left unspecified. The test statistic for testing the model specification is the portmanteau test statistic evaluated at $\hat{\theta}_T$ [Gourieroux, Jasiak (2023)]:

$$\hat{\xi}_T(H) = TL_T(\hat{\theta}_T), \text{ where } L_T(\hat{\theta}_T) = \sum_{h=1}^{H} Tr[\hat{R}_T^2(h, \hat{\theta}_T)], \tag{3.3}$$

and

$$\hat{R}_T^2(h, \theta) = \hat{\Gamma}_T(h; \hat{\theta}_T)\hat{\Gamma}_T(0; \hat{\theta}_T)^{-1}\hat{\Gamma}_T(h; \hat{\theta}_T)'\hat{\Gamma}_T(0; \hat{\theta}_T)^{-1} \tag{3.4}$$

The estimated autocovariances $\hat{\Gamma}_T(h; \hat{\theta}_T)$ are the sample autocovariances of the residuals $\hat{u}_{t,T} = u_t(\hat{\theta}_T) = g(\tilde{Y}_t, \hat{\theta}_T)$ evaluated at the GCov estimator $\hat{\theta}_T$ of θ [6]:

$$\hat{\theta}_T(H) = Argmin_\theta \sum_{h=1}^{H} Tr[\hat{R}_T^2(h, \theta)]. \tag{3.5}$$

Under the null hypothesis of the absence of linear or nonlinear serial dependence, $\hat{\xi}_T(H)$ follows asymptotically the chi-square distribution with degrees of freedom equal to $K^2H - dim\theta$ [see, Gourieroux, Jasiak (2023), Proposition 4]. This result holds only for the GCov estimator $\hat{\theta}_T$.

The GCov estimator is a one-step estimator introduced in Gourieroux, Jasiak (2023):

$$\hat{\theta}_T(H) = Argmin_\theta \sum_{h=1}^{H} Tr[\hat{R}_T^2(h, \theta)]. \tag{3.6}$$

When the model (3.1) is well-specified, the GCov estimator is consistent, asymptotically normally distributed and attains a semi-parametric efficiency bound, under standard regularity conditions [Gourieroux, Jasiak (2023), assumptions A.1, A.2 and Proposition 3].

The Gcov specification test is applicable as a diagnostic tool to a variety of dynamic models, with i.i.d. non-Gaussian errors, including the following:

Example 1: Double Autoregressive Models

i) The Double-Autoregressive (DAR) model [Ling (2007)] is:

[6]They have to be divided by T instead of $(T - H - p)$ to ensure that the sequence of multivariate sample autocovariances remains positive semi-definite.

$$y_t = \phi y_{t-1} + u_t\sqrt{w + \alpha y_{t-1}^2},$$

where $w > 0, \phi \geq 0, \alpha \geq 0$, $\theta = (w, \phi.\alpha)'$ and the u_t's are i.i.d. with a Gaussian or non-Gaussian distribution. We assume that $E(log|\phi + \sqrt{\alpha}u|) < 0$ and the regularity conditions on functions θ, f are satisfied, ensuring the existence of a strictly stationary solution. Then, the semi-parametric representation (3.1) of this model is

$$g(\tilde{y}_t; \theta) = [(y_t - \phi y_{t-1})/\sqrt{w + \alpha y_{t-1}^2} = u_t,$$

This model is strictly stationary for $\phi = 1$ due to the volatility induced stationarity effect.

Example 2: MAR(r,s) Model

The mixed noncausal autoregressive MAR(r,s) process is defined as:

$$(1 - \phi_1 L - \phi_2 L^2 - ... - \phi_r L^r)(1 - \psi_1 L^{-1} - \psi_2 L^{-2} - ... - \psi_s L^{-s})y_t = u_t, \tag{3.7}$$

where the errors are i.i.d., non-Gaussian and such that $E(|u_t|^\delta) < \infty$ for $\delta > 0$. The polynomials $\Phi(L)$ and $\Psi(L)$ in L are of degrees r and s, respectively, with roots strictly outside the unit circle and such that $\Phi(0) = \Psi(0) = 1$. For $r = s = 1$, the MAR(1,1) model is:

$$(1 - \phi L)(1 - \psi L^{-1})y_t = u_t, \tag{3.8}$$

where the parameters ϕ and ψ are two autoregressive coefficients, which are strictly less than one. Coefficient ϕ represents the standard causal persistence, while coefficient ψ depicts non-causal persistence and determines locally explosive patterns and conditional heteroscedasticity. In the MAR(1,1) we have $\theta = (\phi, \psi)'$, $p = 2 = dim(\theta)$ and the semi-parametric representation (3.1) of this model is:

$$g(\tilde{y}_t; \theta) = \Phi(L)\Psi(L^{-1})y_t = u_t$$

Example 3: Causal SVAR Model

The multivariate causal SVAR(p) process is defined by:

$$Y_t = \Phi_1 Y_{t-1} + \cdots + \Phi_p Y_{t-p} + u_t,$$

where $\theta = [vec\Phi_1', ..., vec\Phi_p']'$ and error u_t is a multivariate non-Gaussian serially and cross-sectionally i.i.d. process with finite fourth order moments. The roots of the characteristic equation $det(Id - \Phi_1\lambda - \cdots \Phi_p\lambda^p) = 0$ are of modulus strictly greater than one. We have

$$g(\tilde{Y}_t, \theta) = Y_t - \Phi_1 Y_{t-1} - \cdots - \Phi_p Y_{t-p} = u_t.$$

This model is commonly used in macroeconomic applications.

Example 4: Causal-Noncausal VAR Model

The model is specified as above except that the roots of the characteristic equation $det(Id - \Phi_1\lambda - \cdots \Phi_p\lambda^p) = 0$ are of modulus either strictly greater, or smaller than one and the errors are not assumed cross-sectionally independent. The roots located inside the unit circle create locally explosive patterns and conditional heteroscedasticity, like in the MAR(r,s) model. There exists a unique (strictly) stationary solution (Y_t) with a two-sided $MA(\infty)$ representation, which satisfies model (3.1) with:

$$g(\tilde{Y}_t, \theta) = Y_t - \Phi_1 Y_{t-1} - \cdots - \Phi_p Y_{t-p} = u_t.$$

The causal-noncausal VAR(p) model has been studied in Gourieroux, Jasiak (2017),(2022), Davis, Song (2020) and Swenson (2020). The error u_t of this model cannot be interpreted as an innovation.

3.3 Asymptotic Properties of the GCov Test

This section presents new results on the local alternatives and local power of the GCov specification test to show its validity as a diagnostic test for semi-parametric models with iid non-Gaussian errors. By focusing on the local alternatives, we can study hypotheses on the parameters of a semi-parametric model, given the marginal error distribution. The results below extend the results on the asymptotic distribution of the GCov test statistic derived in Gourieroux and Jasiak (2023).

3.3.1 Null Hypothesis and Asymptotic Size

Let us clarify the definition of the null hypothesis in the semi-parametric framework. As mentioned earlier, there are two types of parameters: vector θ defining the dynamics and functional parameter f defining the error distribution. Then, the theoretical autocovariances $\Gamma(h; \theta, f)$ are functions of both θ and f. The null hypothesis becomes:

$$H_0 : \{\Gamma_0(h) = 0,\ h = 1, ..., H\},$$

where $\Gamma_0(h)$ is the true autocovariance. In terms of parameters θ, f, it corresponds to $\Theta_0 = \{\theta, f : (\Gamma(h; \theta, f) = 0\}$.

So far, the transformation a has not been introduced to simplify the notation. When these transformations are accounted for, the null hypothesis becomes:

$$H_{0,a} \approx \Theta_{0,a} = \{\theta, f : \Gamma_{0,a}(h;\theta, f) = 0,\ h = 1, ..., H\}.$$

When the error terms u_t are serially independent, the nonlinear autocovariances $\Gamma_0(h) = 0,\ \forall h$. Hence, if the model is correctly specified and $H \geq p$, the GCov test of the null hypothesis H_0 tests the absence of linear and nonlinear dependence in the errors u_t. Like in Section 2.2, since one cannot consider all the lags and nonlinear transformations, the null hypothesis of the absence of (non)linear dependence is not equivalent to the independence condition, but is arbitrarily close to it, depending on lag H and the number of nonlinear transformations considered.

The test of the hypothesis H_0 at level α is performed as follows: the null hypothesis H_0 is rejected when $\hat{\xi}_T(H) > \chi^2_{1-\alpha}(K^2H - dim\theta)$ and H_0 is not rejected otherwise. The asymptotic size tends to the nominal size when $T \to \infty$:

$$\lim_{T\to\infty} P_{\theta,f}[\hat{\xi}_T(H) > \chi^2_{1-\alpha}(K^2H - dim\theta)] = \alpha,\ \forall\theta, f \in \Theta_0,\ \forall\alpha.$$

3.3.2 Local Alternatives and Local Power

Rather than evaluating our test against a fixed alternative, we evaluate it against a sequence of local alternatives that drift towards the null at rate $\frac{1}{\sqrt{T}}$. The drifting sequence of alternatives makes it harder and harder to reject the null as the sample grows. The evaluation of the test under several types of local alternatives is hence related to real situations when one has a relatively small amount of data [Davidson, MacKinnon (2006), Section 3, Escanciano (2007), Section 2.3, Dovonon, Hall, Kleibergen (2020), Section 4].

Because the semi-parametric model depends not only on the vector of dynamic parameters θ but also on the functional parameter f, the general alternative hypothesis is difficult to formulate [see Section 4.1 for the discussion of implicit null and alternative hypotheses]. In particular, the alternative could contain models in which the u_t's are serially dependent. Then, the parametrization would have to be enlarged to accommodate these cases. In this section, the transformation a subscripts are again omitted for clarity.

We assume a parametrized (fixed) alternative defined by:

$$H_1^* = \{\theta, \gamma, f : g^*(\tilde{Y}_t;\theta,\gamma) = u_t,\ \text{where } u_t \text{ are i.i.d.,}\} \tag{3.9}$$

with an additional (scalar) parameter γ, a known function g^*, such that $g^*(\tilde{Y}_t;\theta, 0) = g(\tilde{Y}_t;\theta)$ and the dimension $dim(g^*) = dim(u_t)$. Under H_1^*, the autocovariances depend on θ, γ and the marginal distribution f of errors u_t. These autocovariances are denoted by $\Gamma(h;\theta,\gamma, f)$.

The alternative hypothesis is parametrized by θ, γ, f. In terms of the autocovariances (skipping the transformation subscript a), the (fixed) alternative hypothesis is:

$$H_1 = \{\theta, \gamma, f : \Gamma(h; \theta, \gamma, f) = 0,\ h = 1, ..., H\}$$

and the null hypothesis is:

$$H_0 = \{\theta, \gamma, f : \Gamma(h; \theta, 0, f) = 0,\ h = 1, ..., H\} = \{\gamma = 0\}$$

Let us consider the hypothesis testing in the context of a conditionally heteroscedastic model [Dovonon, Hall, Kleibergen (2020)]. The local alternatives concern here the presence of serial dependence at higher lags in 1. the conditional mean of the process, or 2. the conditional variance, or both (3.).

Example DAR(1) model The (fixed) alternatives are for example:

1. $y_t = \phi_1 y_{t-1} + \gamma y_{t-2} + u_t\sqrt{w + \alpha y_{t-1}^2}$
   and $H_{11}^* : u_t = (y_t - \phi_1 y_{t-1} - \gamma y_{t-2})/\sqrt{w + \alpha y_{t-1}^2} = g^*(\tilde{Y}_t; \theta, \gamma)$
2. $y_t = \phi_1 y_{t-1} + u_t\sqrt{w + \alpha y_{t-1}^2 + \gamma y_{t-2}^2}$
   and $H_{12}^* : u_t = (y_t - \phi_1 y_{t-1})/\sqrt{w + \alpha y_{t-1}^2 + \gamma y_{t-2}^2} = g^*(\tilde{Y}_t; \theta, \gamma)$
3. $y_t = \phi_1 y_{t-1} + \gamma y_{t-2} + u_t\sqrt{w + \alpha y_{t-1}^2 + \gamma y_{t-2}^2}$
   and $H_{13}^* : u_t = (y_t - \phi_1 y_{t-1} - \gamma y_{t-2})/\sqrt{w + \alpha y_{t-1}^2 + \gamma y_{t-2}^2} = g^*(\tilde{Y}_t; \theta, \gamma)$
4. $y_t = \phi_1 y_{t-1} + (u_t + \gamma u_{t-1})\sqrt{w + \alpha y_{t-1}^2}$
   and $H_{14}^* : u_t = (\frac{1}{1-\gamma L})\left[(y_t - \phi_1 y_{t-1})/\sqrt{w + \alpha y_{t-1}^2}\right]$ for $\gamma \neq 1$. We see that $u_t \neq g^*(\tilde{Y}_t; \theta, \gamma)$

with $\tilde{Y}_t = Y_{t-1}, ..., Y_{t-p}$ where p is a fixed lag. Therefore, we will not be able to test against this alternative hypothesis.

The local alternatives are defined in a neighborhood of the true value θ_0, f_0 satisfying the null hypothesis. We consider parametric directional alternatives where:

$$\theta_T \approx \theta_0 + \mu/\sqrt{T},\ \ \gamma_T \approx \nu/\sqrt{T},\ \ f_T \approx f_0.$$

Example DAR(1) model The local alternatives of the DAR model obtained by Taylor expansion about $\gamma = 0$ for (fixed) hypotheses 1 to 3 given above are:

1. $y_t = \phi_1 y_{t-1} + \gamma y_{t-2} + u_t\sqrt{w + \alpha y_{t-1}^2}$
   and $H_{L11}^* : u_t = (y_t - \phi_1 y_{t-1} - \gamma y_{t-2})/\sqrt{w + \alpha y_{t-1}^2} = g^*(\tilde{Y}_t; \theta, \gamma)$
2. $y_t \approx \phi_1 y_{t-1} + u_t\left[\sqrt{w + \alpha y_{t-1}^2} + \frac{\gamma}{2}\frac{y_{t-2}^2}{\sqrt{w+\alpha y_{t-1}^2}}\right]$
   and $H_{L12}^* : u_t \approx (y_t - \phi_1 y_{t-1})/\left[\sqrt{w + \alpha y_{t-1}^2} + \frac{\gamma}{2}\frac{y_{t-2}^2}{\sqrt{w+\alpha y_{t-1}^2}}\right] \approx g^*(\tilde{Y}_t; \theta, \gamma)$

3. $y_t \approx \phi_1 y_{t-1} + \gamma y_{t-2} + u_t \left[\sqrt{w + \alpha y_{t-1}^2} + \frac{\gamma}{2} \frac{y_{t-2}^2}{\sqrt{w+\alpha y_{t-1}^2}} \right]$

and $H_{L13}^* : u_t \approx (y_t - \phi_1 y_{t-1} - \gamma y_{t-2}) / \left[\sqrt{w + \alpha y_{t-1}^2} + \frac{\gamma}{2} \frac{y_{t-2}^2}{\sqrt{w+\alpha y_{t-1}^2}} \right]$

$\approx \frac{y_t - \phi_1 y_{t-1}}{\sqrt{w+\alpha y_{t-1}^2}} - \gamma \left[\frac{y_{t-2}}{\sqrt{w+\alpha y_{t-1}^2}} + 0.5 \frac{(y_t - \phi_1 y_{t-1}) y_{t-2}^2}{(w+\alpha y_{t-1}^2)^{3/2}} \right] \approx g^*(\tilde{Y}_t; \theta, \gamma)$

Under the sequence of local alternatives, we consider doubly indexed sequences $(y_{T,t})$, i.e. a sequence of processes indexed by T (triangular array). In this framework, what matters is the local impact on the autocovariances, i.e.:

$$\Gamma(h; \theta_T, \gamma_T, f_0) \approx \Gamma(h; \theta_0, 0, f_0) + \frac{\partial \Gamma(h; \theta_0, 0, f_0)}{\partial \theta'} (\theta_T - \theta_0) + \frac{\partial \Gamma(h; \theta_0, 0, f_0)}{\partial \gamma'} (\gamma_T - \gamma_0),$$

with $\Gamma(h; \theta_0, 0, f_0) = 0$. This leads to a local alternative written on the autocovariance:

$$\Gamma(h; \theta_T, \gamma_T, f_0) = \Delta(h; \theta_0, f_0, \mu, \nu) / \sqrt{T} \tag{3.10}$$

with

$$\Delta(h; \theta_0, f_0, \mu, \nu) = \frac{\partial \Gamma(h; \theta_0, 0, f_0)}{\partial \theta'} \mu + \frac{\partial \Gamma(h; \theta_0, 0, f_0)}{\partial \gamma'} \nu. \tag{3.11}$$

Its vec representation is denoted by $\delta(h; \theta_0, f_0, \mu, \nu) = vec\Delta(h; \theta_0, f_0, \mu, \nu)$.
Then, the asymptotic local power of the test, given f_0 fixed, is

$$\lim_{T \to \infty} P_{\theta_T, \gamma_T, f_0} [\hat{\xi}_T(H) > \chi^2_{1-\alpha}(K^2 H - dim\theta)] = \beta(\theta_0, f_0, \mu, \nu; \alpha),$$

for any μ, ν, α and $(\theta_0, f_0) \in \Theta_0$.

Proposition 1: Under the sequence of local alternatives, and the regularity conditions of Appendix B,
i) The autocovariance estimator

$$\hat{\Gamma}_T(h; \theta) = \frac{1}{T} \sum_{t=1}^{T} g(y_{T,t}; \theta) g'(y_{T,t-h}; \theta) - \frac{1}{T} \sum_{t=1}^{T} g(y_{T,t}; \theta)' \frac{1}{T} \sum_{t=1}^{T} g(y_{T,t}; \theta)$$

converges in probability to

$$\hat{\Gamma}_T(h; \theta) \to \Gamma(h; \theta_0, 0, f_0, \theta),$$

for all θ, h where the limit is computed under the null hypothesis. This convergence is uniform in θ.
ii) The GCov estimator converges in probability to θ_0:

$$\hat{\theta}_T \to \theta_0$$

Proof: The proof is based on the Law of Large Numbers (LLN) for doubly indexed sequences [see e.g. Andrews (1988), Newey (1991) and Appendix B for regularity conditions]. The LLN implies the convergence of the estimated autocovariances.

Let us now discuss the convergence of the GCov estimator. The objective function:

$$L_T(\theta) = \sum_{h=1}^{H} Tr[\hat{\Gamma}_T(h;\theta)\hat{\Gamma}_T(0;\theta)^{-1}\hat{\Gamma}_T(h;\theta)'\hat{\Gamma}_T(0;\theta)^{-1}] \tag{3.12}$$

tends to

$$L_\infty(\theta) = \sum_{h=1}^{H} Tr[\Gamma(h;\theta_0,0,f_0,\theta)\Gamma(0;\theta_0,0,f_0,\theta)^{-1}\Gamma(h;\theta_0,0,f_0,\theta)'\Gamma(0;\theta_0,0,f_0,\theta)^{-1}]. \tag{3.13}$$

This limit is the same as the limit of the objective function under the null hypothesis. Then, the consistency is proven as in Gourieroux, Jasiak (2023).

Let us show that the test statistic converges in distribution under local alternatives to a non-central chi-square distributed variable with a non-centrality parameter involving the direction of the perturbation. Below, we derive the distribution of the test statistic computed from the residuals of the model $g(y_t,\theta) = u_t$ estimated by the GCov estimator under the sequence of local alternatives. The proof is given in Appendix B and based on the Central Limit Theorem (CLT) for doubly indexed sequences [see e.g. Wooldridge, White (1988) and Appendix B].

Proposition 2: Let us consider the specification test of the null hypothesis:

$$H_0 = \Theta_0 = \{\theta, f : \Gamma(h;\theta,f) = 0 \ \ \forall h = 1,...,H\},$$

against the sequence of local alternatives:

$$H_{1,T} = \Theta_{1,T} = \{\theta = \theta_0 + \mu/\sqrt{T}, \gamma = \nu/\sqrt{T}, f = f_0, \text{ with } (\theta_0, f_0) \in \Theta_0\}.$$

The expansion of the test statistic under the sequence of local alternatives is:

$$\hat{\xi}_T(H) = T\sum_{h=1}^{H}\{vec[\sqrt{T}\hat{\Gamma}_T(h;\theta_T,\gamma_T,f_0)]\Pi(h;\theta_0,f_0)vec[\sqrt{T}\hat{\Gamma}_T(h;\theta_T,\gamma_T,f_0)]\} + o_p(1), \tag{3.14}$$

where

$$\begin{aligned}\Pi(h;\theta_0,f_0) &= [\Gamma_0(0,\theta_0,f_0)^{-1}\otimes\Gamma_0(0,\theta_0,f_0)^{-1}] - [\Gamma_0(0,\theta_0,f_0)^{-1}\otimes\Gamma_0(0,\theta_0,f_0)^{-1}]\frac{\partial vec\Gamma(h,\theta_0,f_0)}{\partial\theta'} \\ &\quad \left\{\frac{\partial vec\Gamma(h,\theta_0,f_0)'}{\partial\theta}[\Gamma_0(0,\theta_0,f_0)^{-1}\otimes\Gamma_0(0,\theta_0,f_0)^{-1}]\frac{\partial vec\Gamma(h,\theta_0,f_0)}{\partial\theta}\right\}^{-1} \\ &\quad \times\frac{\partial vec\Gamma(h,\theta_0,f_0)'}{\partial\theta'}[\Gamma_0(0,\theta_0,f_0)^{-1}\otimes\Gamma_0(0,\theta_0,f_0)^{-1}].\end{aligned}$$

Then, under the sequence of local alternatives, $\hat{\xi}_T(H) \overset{a}{\sim} \chi^2(K^2H - dim\theta, \lambda(\theta_0, f_0, \mu, \nu))$, where the non-centrality parameter is

$$\lambda(\theta_0,f_0,\mu,\nu) = \sum_{h=1}^{H}\delta(h;\theta_0,f_0,\mu,\nu)'\Pi(h;\theta_0,f_0)\delta(h;\theta_0,f_0,\mu,\nu)),$$

with $\delta(h;\theta_0,f_0,\mu,\nu) = \frac{\partial\Gamma(h;\theta_0,0,f_0)}{\partial\theta'}\mu + \frac{\partial\Gamma(h,\theta_0,0,f_0)'}{\partial\gamma'}\nu$.
Proof: The proof of Proposition 2 is given in Appendix B.

Let the cumulative distribution function (c.d.f.) of the non-central chi-square distribution be denoted by $F(x;\kappa,\lambda)$, where κ denotes the degrees of freedom and λ is the non-centrality parameter. Moreover, $F(x;\kappa,\lambda) = 1 - Q_{\kappa/2}(\sqrt{\lambda},\sqrt{x})$, where $Q_\delta(a,b)$ is a Marcum Q-function. For positive integer values of δ it is defined as:

$$Q_\delta(a,b) = \begin{cases} H_\delta(a,b) & a<b, \\ 0.5 + H_\delta(a,a), & a=b \\ 1 + H_\delta(a,b), & a>b \end{cases}$$

where $H_\delta(a,b) = \frac{\zeta^{1-\delta}}{2\pi}exp(-\frac{a^2+b^2}{2})\int_0^{2\pi}\frac{cos(\delta-1)w - \zeta cos\,\delta\, w}{1-2\zeta cosw+\zeta^2}exp(ab\cos w)dw$ and $\zeta = a/b$. Then, from Proposition 1, it follows that the local asymptotic power is given by:

$$\beta(\theta_0,f_0,\mu,\nu;\alpha) = Q_{(K^2H-dim\theta)/2}[\sqrt{\lambda(\theta_0,f_0,\mu,\nu)}, \sqrt{\chi^2_{1-\alpha}(K^2H-dim\theta)}].$$

From the monotonicity property of the Q-function, it follows that the local asymptotic power function is a strictly decreasing function of the non-centrality parameter.

4 Increasing the Set of Autocovariance Conditions

In this section we discuss how the asymptotic performance of the GCov portmanteau test of the independence hypothesis can be improved by increasing the number of nonlinear transformations, and possibly by taking into account an infinite number of autocovariance conditions.

4.1 Implicit Null and Alternative Hypotheses

The GCov estimator has been applied so far using a finite set of zero unconditional autocovariance conditions, and given an autoregressive order H and a set of K nonlinear error transformations. If this set allows us to identify parameter θ, then the test is necessarily asymptotically locally the most powerful for the associated implicit null and alternative hypotheses.

At this point, it is important to compare the implicit null and alternative hypotheses considered with those of potential interest. More specifically, we could consider the following hypotheses that concern the distribution f of the process (u_t):

i) $H_{0,\mathcal{A}} = \{f : Cov[a(u_t), \alpha(u_{t-h})] = 0, \ \forall h, \ \forall a, \alpha \in \mathcal{A}\}$ and also the associated alternative $H_{1,\mathcal{A}}$. Those hypotheses depend on the selected set $\mathcal{A}$ of transformations.

ii) $H_{0,pair} = \{f : u_t,$ and u_{t-h} are independent, $\forall h\}$ and its alternative $H_{1,pair}$.

iii) $H_{0,ind} = \{f : u_t, u_{t-1}, ..., u_{t-h}$ are independent $\forall t, h\}$ and its alternative $H_{1,ind}$.

iv) $H_{0,iid} = \{f : u_t' s$ are iid$\}$ and its alternative $H_{1,iid}$.

We have $H_{0,\mathcal{A}} \supset H_{0,pair} \supset H_{0,ind} \supset H_{0,iid}$ and $H_{1,\mathcal{A}} \subset H_{1,pair} \subset H_{1,ind} \subset H_{1,iid}$.

The test of $H_{0,\mathcal{A}}$ is consistent and of asymptotic power of 1 against $H_{1,\mathcal{A}}$. However, this test is not of asymptotic power 1 for $H_{1,pair}, H_{1,ind}$ and $H_{1,iid}$. By increasing the set of nonlinear transformations a in $\mathcal{A}$, we hope to increase the set of alternatives against which the asymptotic power of the test is 1.

4.2 The Semi-Parametric Efficiency Bound

The asymptotic performance of the GCov portmanteau test is linked to the asymptotic properties of the GCov estimator of parameter θ. Let us now distinguish the semi-parametric models and the associated semi-parametric efficiency bounds, which are:

i) The semi-parametric efficiency bound that accounts for the pairwise independence between u_t and u_{t-h} for any h.

ii) The semi-parametric efficiency bound that takes into account the joint independence of $u_t, u_{t-1}, ..., u_{t-h}$, since the pairwise independence does not imply the joint independence.

iii) The semi-parametric efficiency bound that also takes into account the fact that the u_t's are identically distributed. The example given below shows that this efficiency bound coincides with the parametric efficiency bound.

In the framework of nonlinear autocovariance-based test statistics with a one-step GCov estimator, we could attain only the first semi-parametric efficiency bound i), but not the parametric efficiency bound under $H_{0,iid}$, which requires a two-step approach as shown in the example below.

Example: Adaptive Estimator

Let $\hat{\theta}_T$ denote a GCov estimator based on a finite set $\mathcal{A}$ of nonlinear transformations. Given this consistent estimator, we can compute the residuals $\hat{u}_{T,t}, t = 1, ..., T$ of the model and then estimate nonparametrically the true density f_0 of u_t by the kernel-based density $\hat{f}_T(u)$ based on the residuals $\hat{u}_{T,t}, t = 1, ..., T$. Next, in the second step, parameter θ can be estimated by a pseudo-maximum likelihood method, where the true density f_0 is replaced by $\hat{f}_T(u)$ [see e.g. Bickel (1982), Newey (1988)]. Under standard regularity conditions, this leads to a two-step estimator that reaches the parametric efficiency bound. Because of this improvement of the asymptotic properties of the GCov estimator, the pseudo-likelihood ratio test based on this two-step estimator will have better asymptotic power properties.

4.3 How to Choose an Infinite Set of Transformations

The asymptotic properties of the GCov estimator and of the associated test statistics can be improved by increasing the finite set of nonlinear zero autocovariance conditions to a larger finite, or infinite set.

The extension to an infinite set of nonlinear autocovariance conditions is easy when these conditions correspond to an orthonormal basis of the Hilbert space $L^2(u_t, u_{t-1}, ...)$. In our framework, we can increase the set of nonlinear autocovariance conditions either by increasing the maximum lag H, or the set of transformations. We saw that, under the null hypothesis, the orthogonality of nonlinear autocovariance conditions with respect to the lag is satisfied. This explains the simplified form of the test statistic written as a sum of terms associated with lags $h = 1, ..., H$.

In contrast, the set of nonlinear transformations does not necessarily correspond to an orthonormal basis, and inverting the variance matrix of a large dimension can become a problem from both the theoretical and computational perspectives. The standard approach consists in considering a (large) infinite set of square-integrable transformations of $u_t, ...u_{t-h}$, and orthonormalizing them progressively[7]. This requires:

1: Considering an infinite set of transformations that allows for identifying the unknown distribution f_0 of u_t.

2: Ranking the transformations in a countable sequence of transformations and orthonormalizing it in the Hilbert space. The ranking has to be done carefully to avoid any information

[7]Note that it is always possible to get an orthonormal basis of transformations of u_t, u_{t-h} by considering the product of an orthonormal basis of u_t and a (possibly the same) orthonormal basis of u_{t-h}.

loss. It is facilitated if the set of transformations admits a countable dense subset [see, e.g. Bierens (1990), Corollary 1, for such a subset of exponential transforms, Royden (2010) for orthonormalization in the Hilbert space].

Let us now discuss how to select the basis of transformations (in addition to u_t and the transformation u_t^2 that can be informative of some parameters). An important feature of the causal-noncausal models considered in our paper are the extreme risks and their persistence that create the local explosive patterns, including the bubbles. This implies that the error does not necessarily have power moments. Moreover, some of the parameters driving those extreme risks, may be non-identifiable from the transformations u_t, u_t^2 only. In addition, we may have to disentangle the negative and positive extreme risks.

In this respect, some standard linear systems of generators are not convenient. For example: i) the polynomial transformations of u_t cannot be used if u_t has no moments of order greater than 3; ii) The sine-cosine transformations are not informative of the tail parameters [see e.g. Wan and Davis (2022), Fokianos and Pitsillou (2018) for a test of the independence hypothesis based on a joint characteristic function]. The same remark applies to the standard bases of splines.

A natural choice is a system of generators that assigns weights to the standard polynomials with the selected weights ensuring their square integrability. Let us discuss how to choose these weights. For ease of exposition, let us consider a process with positive errors u_t, and with trajectories admitting positive and increasing bubbles only caused by a large positive shock to u_t. Let us also consider a univariate process[8]. Then, a linear system of generators is:

$\mathcal{A} = \{a_{t,p}(u) = u^p \exp(-tu),\ p \in \mathcal{N},\ t \in [0,1]\}$,

with a countable dense subsystem given by:

$\mathcal{A}_n = \{a_{t_j,n}(u) = u^p \exp(-t_{j,n}u),\ p \in \mathcal{N},\ t_{j,n} \in [0,1],\ j = 1, ...n\}$,

and $(t_{1,n}, ..., t_{j,n})$ which is dense in $[0,1]$ when n tends to infinity [see Bierens (1990), page 1448, for a similar approach for a bounded variable u and generators $a_n(x) = \alpha_{n,0} + \sum_{j=1}^n \alpha_{n,j} \exp(t_j' u)$.].

The limiting system $\mathcal{A}_n$ with $n \to \infty$ allows for identifying the distribution of u, which follows from the proposition below, with the inversion formula of the real Laplace transform called the Post's inversion formula [Post (1930), Feller (1971), Chapter 13, for the modern proof].

[8]The extension to a multivariate u_t is performed by crossing the generators of univariate u_t's (see, Section 4.4).

Proposition 3: The System of Generators

Let us assume a distribution of U with continuous density $f_0(u)$ such that:

$\sup_{u>0} f_0(u)/\exp(bu) < \infty$, for some $b > 0$.

i) The distribution of U is characterized by the knowledge of the Laplace transform: $\Psi(t) = E[\exp(-tU)]$, $t \in [0,1]$.

ii) A system of generators is $\mathcal{A}^* = \{a_t(u) = \exp(-tu),\ t \in [0,1]\}$.

iii) The Post's inversion formula is:

$$f_0(v) = \lim_{n\to\infty} \frac{1}{n!} \left(\frac{n}{v}\right)^{n+1} E[U^n \exp\left(-\frac{n}{v}U\right)],\ \forall v\ a.e.$$

Note that $\mathcal{A}^* = \{a_t(u) = \exp(-tu),\ \mathcal{A} \in [0,1]\}$ is not necessarily a convenient system of generators, since we do not know if f_0 can be found from the limit of combinations of decreasing exponentials only. The Post's inversion formula shows that the products of exponentials and polynomials have to be considered. Then, a linear system of generators is:

$$\mathcal{A} = \{a_{t,p}(u) = u^p \exp(-tu),\ p \in \mathbb{N}, t \in [0,1]\},$$

with a countable dense subsystem given by:

$$\mathcal{A}_n = \{a_{t_{j,n},p}(u) = u^p \exp(-t_{j,n}u),\ p \in \mathbb{N}, t_{j,n} \in [0,1],\ j = 1, ..., n\},$$

and $(t_{1,n}, ..., t_{n,n})$ is dense in $[0,1]$ when n tends to infinity [see, Bierens (1991), page 1448, for a similar approach].

The condition imposed on the density function means that the distribution cannot have a large mass at zero, but its right tail can be of any magnitude.

From the above Proposition it follows that an appropriate weighting is ensured by the decreasing exponential transforms, which are introduced to provide square integrability of the power transforms. In fact, only the weights with t close to 0 are useful. To see that, let us recall that this real Laplace transform Ψ of a positive variable is characterized by its Taylor series expansion:

$$E[\exp(-tU)] = \sum_{j=0}^{\infty} \frac{t^j}{j!}\mu_j.$$

If U admits power moments of any order, we have $\mu_j = E(U^j)$, $\forall j$. In our framework, such moments may not exist, but regardless this series expansion exists and Taylor's coefficients μ_3, μ_4 define new notions of skewness and kurtosis measures.

Remark 1: In practice, the Post's inversion formula does not provide a tractable mean of inverting the Laplace transform and it is an ill-posed problem [see the remark in the conclusion of Bryan (2006)]. In this respect, the GCov estimator can be seen as an approach for solving this ill-posed problem when the objective is to estimate θ. In our test framework, the Post's inversion formula is used only to show that $\mathcal{A}$ is an adequate linear system of generators.

Remark 2: Similar results can be obtained if additional information is available on the magnitude of the tail of the distribution of U, for example, if the tails are Pareto. Then, we can apply the Hardy-Littlewood Tauberian theorem with a Pareto weighting function.

Remark 3: If the error u_t takes both positive and negative values, we can observe "positive" and "negative" bubbles. Then, the weights have to be replaced by $\exp(-t|U|)$, and the powers of U have to be allowed to take positive and also negative signs for odd powers. If u_t has a symmetric distribution, the generators will be $a_{t,p} = |u|^p \exp(-t|u|)$.

In practice, it may be better to limit the number of transformations and choose the informative transformations for the estimation keeping in mind that the parameters that are difficult to identify can be revealed either by well-chosen transformations, or by some standard transformations by increasing the lags H to capture the persistence properties. It may be useful to consider quadratic transformations along with some decreasing exponential transformations.

4.4 The Orthonormalization of the System of Generators

It is not possible to construct an exact orthonormal basis from a system of generators since the true distribution of U under the i.i.d. hypothesis is unknown. In fact, a two-step approach is required. We derive below the orthonormalization step in the multidimensional case and discuss the form of the portmanteau statistic in the following subsection. The steps are the following:

step 1. Consider a finite set $\mathcal{A}_0$ of transformations from which θ is identifiable. Estimate θ by the GCov estimator $\hat{\theta}_T$ and compute $\hat{u}_{T,t},\ t = 1, ..., T$.

step 2. Consider an infinite set $\mathcal{A} = \{a(u) = u_1^{p_1}, ..., u_J^{p_J} \exp(-\tau' u),\ p_1, ..., p_J \in \mathcal{N},\ \tau \in [0,1]^J\}$

and the sequence of finite sets $\mathcal{A}_n$:

$$\mathcal{A}_n = \{a(u) = u_1^{p_1}, ..., u_J^{p_J} \exp(-\tau'_{j,n} u),\ p_1, ..., p_J \in (0, 1, ..., P_n),\ \tau_{1,n}, ..., \tau_{n,n} \in [0,1]^J\},$$

where $(\tau_{1,n}, ..., \tau_{n,n})$ is dense in $[0,1]^J$ when $n \to \infty$.

The number of transformations in $\mathcal{A}_n$ is $P_n^J J^n$, where $K_n \equiv P_n^J J^n$ with the elements $a_{k,n}, k = 1, ..., K_n$ ranked in an increasing order:

$\mathcal{A}_n = \{a_{k,n},\ k = 1, ..., K_n = P_n^J J^n\}.$

step 3. Orthonormalization (Hilbert-Schmidt)

$a_{k,n},\ k = 1, ..., P_n^J J^n$ need to be mapped into an orthonormal basis of $a^*_{k,n},\ k = 1, ..., K_n$ as follows:

We run recursion to get at step m the orthonormal functions $a^*_{k,n},\ k = 1, ..., m$, with zero mean.

a) The starting point is the following:

Regress $a_{1,n}(\hat{u}_{T,t})$ on the constant:

$$a_{1,n}(\hat{u}_{T,t}) = \alpha_{1,n,T} + \hat{w}_{1,n,T,t},$$

$t = 1, ..., T$ with the residual $\hat{w}_{1,n,T,t}$. Next, compute:

$$a^*_{1,n,T}(u) = \hat{w}_{1,n,T}(u)/||\hat{w}_{1,n,T}||_T,$$

where $\hat{w}_{1,n,T}(u) = a_{1,n}(u) - \alpha_{1,n,T}$. Let $R^2_{1,n,T}$ denote the R-square in the above regression. Then $a^*_{1,n,T}$ is used if $1 - R^2_{1,n,T}$ is sufficiently different from 0, i.e. $1 - R^2_{1,n,T} > \epsilon_{n,T}$, where $\epsilon_{n,T}$ is a tuning parameter that needs to be appropriately chosen. Otherwise, start from $a_{2,n}$.

b) Next, $a_{2,n}$ is projected on $a^*_{1,n,T}$ as follows:

We run the regression:

$$a_{2,n}(\hat{u}_{T,t}) = \alpha_{2,n,T} + \beta_{2,n,T} a^*_{1,n,T}(\hat{u}_{T,t}) + \hat{w}_{2,n,T,t},$$

$t = 1, ..., T$ with the residuals $\hat{w}_{2,n,T,t}$. Next, compute:

$$a^*_{2,n,T}(u) = \hat{w}_{2,n,T}(u)/||\hat{w}_{2,n,T}||_T,$$

where $\hat{w}_{2,n,T}(u) = a_{2,n}(u) - \alpha_{2,n,T} - \beta_{2,n,T} a^*_{1,n,T}(u)$. This is possible if $||\hat{w}_{2,n,T}||_T$ is sufficiently different from 0, that arises if $1 - R^2_{2,n,T} > \epsilon_{n,T}$. Otherwise, apply to $a_{3,n}$ [see, Bierens (1990), page 1448 for an analogous approach].

c) The third step is

$$a_{3,n}(\hat{u}_{T,t}) = \alpha_{3,n,T} + \beta_{3,1,n,T} a^*_{1,n,T}(\hat{u}_{T,t}) + \beta_{3,2,n,T} a^*_{2,n,T}(\hat{u}_{T,t}) + \hat{w}_{3,n,T,t},$$

and so on. Since the regressors are orthogonal, the multivariate regressions can be replaced by simple regressions in practice.

At the end, we get a set of transformations $a^*_{k,n}, k = 1, ..., K_n$, which are zero mean and orthonormal with respect to the sample distributions of $\hat{u}_{T,t},\ t = 1, ..., T$.
Since the autocovariance conditions concern pairs u_t, u_{t-h} and transformations of the type $a(u_t), \alpha(u_{t-h})$, we will have to consider transformations in $\mathcal{A}^2_n$, where $dim(\mathcal{A}^2_n) = P_n^{2J} J^{2n} = K_n^2$.

4.5 The Two-step Portmanteau Statistic and its Asymptotic Distribution

The literature on the method of moments and the associated over-identification test statistic with an infinite number of moments considers, in general, the unweighted rather than optimally weighted objective functions [see e.g. Han and Phillips (2006), Fokianos and Pitsillou (2018), Escanciano (2007), Wan and Davis (2022)]. The orthonormalization approach outlined in Section 4.4 is a solution to solve this difficulty. Indeed, after the orthonormalization, the objective function has the following expression:

$$\xi_{n,T}(\theta) = \sum_{h=1}^{H} \sum_{j=1}^{K_n} \sum_{k=1}^{K_n} \left(\frac{1}{T} \sum_{t=1+h}^{T} a^*_{j,n,T}[g(\tilde{y}_t; \theta)] a^*_{k,n,T}[g(\tilde{y}_{t-h}; \theta)] \right)^2,$$

in which the inverse of the variance-covariance matrix is equal to the identity matrix.

At this point, different test statistics can be considered [see the discussion in Han and Phillips (2006), Section 5], such as:

$$\hat{\xi}_{n,T} = \xi_{n,T}(\hat{\theta}_T),$$

evaluated at the first-step GCov estimator, or

$$\hat{\hat{\xi}}_{n,T} = \xi_{n,T}(\tilde{\theta}_{n,T}), \text{ with } \tilde{\theta}_{n,T} = Argmin_\theta \, \xi_{n,T}(\theta),$$

evaluated at the two-step GCov estimator.

We consider below the two-step portmanteau statistic $\hat{\hat{\xi}}_{n,T}$. The asymptotic behavior of this statistic is derived in two steps [see e.g. Koenker, Machado (1999)]:
step 1: We consider the infeasible portmanteau test statistic:

$$\xi^0_{n,T} = \sum_{h=1}^{H} \sum_{j=1}^{K_n} \sum_{k=1}^{K_n} \left(\frac{1}{T} \sum_{t=1+h}^{T} a^*_{j,n}[g(\tilde{y}_t; \theta_0)] a^*_{k,n}[g(\tilde{y}_{t-h}; \theta_0)] \right)^2.$$

where $a^*_{k,n}$ are the transformations obtained by a normalization with the true distribution f_0 and θ replaced by θ_0.

step 2: Next, the uncertainty due to the estimation of θ_0 and f_0 is taken into account.

i) Let us assume that u_t has non-negative components, a continuous density $f_0(u)$ that satisfies a tail condition:

$$\sup_{u>0} f_0(u)/\exp(b'u) < \infty, \text{ for some } b,$$

with strictly positive components, ensuring that the transformations in $\mathcal{A}_n$ are uniformly integrable. Then, under the hypothesis $H_{0,iid}$ of i.i.d. $u_t = g(\tilde{y}_t, \theta_0)$, for any fixed K_n, the multidimensional Lindeberg-Feller condition for the convergence in distribution of

$$\frac{1}{T}\sum_{t=1+h}^{T} a^*_{j,n}[u_t]a^*_{k,n}[u_{t-h}],\ j,k = 1,...,K_n,\ h = 1,...,H.$$

to a standard normal $N(0, Id_{K_n^2 H})$ is satisfied.
Then, under an additional tightness condition when K_n tends to infinity, we deduce that

$$(\xi^0_{n,T} - HK_n^2)/\sqrt{2HK_n^2} \xrightarrow{d} N(0,1),$$

when T and K_n tend to infinity.

ii) This asymptotic behavior is also satisfied for the two-step statistic $\hat{\xi}_{n,T} = \xi_{n,T}(\tilde{\theta}_T)$ under $H_{0,iid}$:

$$(\hat{\xi}_{n,T} - HK_n^2)/\sqrt{2HK_n^2} \xrightarrow{d} N(0,1),$$

provided that: a) the first-step estimator $\hat{\theta}_T$ is consistent, asymptotically Normally distributed; b) the regularization tuning parameter $\epsilon_{n,T}$ tends to zero at an adequate rate; c) T and K_n tend to infinity at a rate such that K_n^2/T tends to zero,
These conditions ensure that the uncertainty due to the approximation of θ_0 by $\hat{\theta}_T$ and of the true orthonormalization by the estimated one are negligible [see e.g. Donovon and Gospodinov (2024) for a similar argument].

As mentioned earlier, our discussion is focused on the increase of the set $\mathcal{A}_n$ of transformations. It could be possible to also increase the maximum lag H with the number of observations and get a similar result when T, H, K_n tend to infinity at a rate such that $(HK_n^2)/T$ tends to zero.

The asymptotic results given above are valid for any choice of sequence $\mathcal{A}_n$ and the ranking of transformations in $\mathcal{A}_n$. However, the choice of sequence $\mathcal{A}_n$ and of transformation ranking will have an effect in finite sample.

5 Bootstrap-Adjusted GCov Test

This section describes the bootstrap-adjusted GCov test of a multivariate causal nonlinear dynamic model and provides the regularity conditions ensuring its validity. The application of the bootstrap GCov test to noncausal processes and the bootstrap GCov test based on the AML estimator are illustrated through simulations in Section 6.3.

5.1 Bootstrap Under the Null Hypothesis

Let us now introduce a bootstrap-adjusted GCov test that can be used to correct for the finite sample bias of the GCov estimator, by providing an asymptotically valid critical value found by bootstrap.

More specifically, to approximate the distribution of bootstrapped $\tilde{\xi}_T$, we compute the test statistic $\xi(H, \hat{\theta}_T^s, \hat{f}_T^s)$ with $\hat{\theta}_T^s$ and $\hat{f}_T^s$ both obtained from the bootstrapped residuals, and the bootstrapped values $y_1^s, ..., y_T^s$. We assume $\tilde{Y}_t = (Y_t, Y_{t-1}, ..., Y_{t-p}) \equiv \underline{Y_t}$ and denote by c the inverse of function g with respect to y_t. Then, the critical value of the test statistic $\tilde{\xi}_T(H)$ can be found by bootstrap along the following steps:

1. Draw randomly with replacements T residuals $\hat{u}_{T,t}^s, t = 1, ..., T$ from residuals $\hat{u}_{T,t} = g(\bar{Y}_t, \hat{\theta}_T), t = 1, ..., T$.

2. Build the bootstrapped time series of length T: $y_{T,t}^s = c(\underline{\tilde{y}_{T,t-1}^s}, \hat{u}_{T,t}^s, \hat{\theta}_T), t = 1, ..., T, \ s = 1, ..., S$ [9].

3. Re-estimate the model parameter vector θ by GCov from $y_{T,t}^s, \ t = 1, ..., T$, providing $\hat{\theta}_T^s, \ s = 1, ..., S$.

Then, under the regularity conditions discussed in Section 5.4, the asymptotic distribution of $\sqrt{T}(\hat{\theta}_T^s - \hat{\theta}_T)$ conditional on the sample $y_t, t = 1, ..., T$ is the same as the asymptotic distribution of $\sqrt{T}(\hat{\theta}_T - \theta_0)$. In particular, the bootstrap tests of hypotheses on θ are asymptotically valid under the null.

4. Compute the test statistic $\hat{\xi}_T^s(H)$ from $\hat{u}_{T,t}^s, t = 1, ..., T$, their nonlinear transforms and $\hat{\theta}_T^s$.

5. Rank the test statistics $\hat{\xi}_T^s(H), \ s = 1, ..., S$, and use the 95th percentile $\hat{q}_{T,95\%}$, say, as the critical value for testing the null hypothesis of absence of nonlinear or linear dependence. Then, the null hypothesis is rejected if:

[9] If the process is a mixed VAR, the bootstrapped values $y_1^s, ..., y_T^s$ can be computed from $\hat{u}_{T,t}^s$ using the formulas in Gourieroux, Jasiak (2017). For a VAR model in a multiplicative representation, see Lanne, Saikkonen (2013). For univariate MAR(r,s) processes, see Gourieroux, Jasiak (2016).

$$\hat{\xi}_T > \hat{q}_{T,95\%},$$

and it is not rejected, otherwise.

5.2 An Extended Continuous Updating GMM

To explain the relation with the current literature on bootstrap applied to overidentification tests based on either moment or covariance conditions, it is interesting to consider an alternative Continuous Updating GMM (CUGMM) that allows for mean adjustment of each transformation in the GCov context.

For ease of exposition, let us assume $H = 1$. To establish this relationship, we need to introduce an alternative set of moment conditions and an extended set of parameters. More precisely, let the parameters be denoted by θ, γ, where $dim(\gamma) = J$. The moment conditions are:

$$\begin{aligned} E[a_k[g(\tilde{Y}_t, \theta)] - \gamma_k] &= 0,\ k = 1, ..., K, \\ E\{[a_j[g(\tilde{Y}_t, \theta)] - \gamma_j][a_k[g(\tilde{Y}_{t-1}, \theta)] - \gamma_k]\} &= 0,\ j, k = 1, ..., K. \end{aligned} \tag{5.1}$$

Thus, we introduce a first set of conditions that jointly identify the additional set of parameters.

Proposition 4:

i) Under the condition of just-identification, $dim(\theta) = K^2$, the CUGMM applied to system (5.1) leads to estimators $\hat{\theta}_T, \hat{\gamma}_T$ and optimal objective function $\hat{\xi}_T^*(\hat{\theta}_T, \hat{\gamma}_T)$ such that : $\hat{\theta}_T$ is equal to the GCov estimator of θ, $\hat{\xi}_T^*(\hat{\theta}_T, \hat{\gamma}_T)$ is equal to the optimal GCov objective function.

ii) Under overidentification, $dim(\theta) < K^2$, the GCov and CUGMM estimators of θ differ at order $1/T$. The associated test statistics have the same asymptotic chi-square distributions at the first order, but differ at higher orders.

Proof: i) If $dim(\theta) = K^2$, due to the just identification of γ for a given θ, we can concentrate the CUGMM objective function with respect to γ to get:

$$\hat{\gamma}_T(\theta) = \frac{1}{T} \sum_{t=1}^{T} a[g(\tilde{Y}_t, \theta)].$$

Next, by plugging in $\hat{\gamma}_T(\theta)$ into the CUGMM objective function, we find that the CUGMM objective function concentrated with respect to γ is equal to the GCov objective function. The result follows.

ii) Under overidentification, the mean adjustments underlying the GCov and CUGMM estimators differ at order $1/T$ as it is easy to see from the asymptotic Taylor expansions of the estimators, that is by their biases at order $1/T$.

QED.

Since the GCov and extended CUGMM estimators differ asymptotically by the bias at order $1/T$, and the bootstrap methods applied to the above estimators are expected to adjust for these biases, sufficient regularity conditions for the validity of bootstrap for the GCov and extended CUGMM, respectively, will be the same.

Remark 4: The effect of demeaning at order $1/T$ is easy to show when estimating an autoregressive coefficient in the AR(1) model: $y_t - \rho y_{t-1} = u_t$, where u_t are i.i.d., with mean 0. Two OLS-type estimators can be considered:

$\hat{\rho}_T = \sum_{t=2}^{T}(y_t - \bar{y}_T)(y_{t-1} - \bar{y}_T)/\sum_{t=2}^{T}(y_{t-1} - \bar{y}_T)^2$, with $\bar{y}_T = \frac{1}{T}\sum_{t=1}^{T} y_t$, and

$\tilde{\rho}_T = \sum_{t=2}^{T}(y_t - \bar{y}_T)(y_{t-1} - \tilde{y}_{T-1})/\sum_{t=2}^{T}(y_{t-1} - \tilde{y}_{T-1})^2$, with $\tilde{y}_{T-1} = \frac{1}{T-1}\sum_{t=2}^{T} y_t$,

The mean adjustments differ at order $1/T$.

Remark 5: To illustrate point ii) of the proof, the relationship between GCov and CUGMM can be used to reinterpret the problem of increasing the set of transformations of Section 4 in terms of GMM. Whereas the dimension of the parameter $dim(\theta)$ is fixed when the set of transformations increases, the dimension of parameter θ, γ will increase from the GMM perspective and become infinite with an infinite number of transforms.

5.3 Bootstrap Based on GMM or Covariance Estimators

There exists a large literature on bootstrap applied to GMM estimators and the associated overidentification test statistics. This literature considers the same type of portmanteau statistics, but the approaches differ with respect to the variables that are initially resampled, the assumptions on the observations, and the selected moment or covariance conditions.

i) A part of this literature considers a direct resampling of the observations themselves, usually when the moment conditions are of the type:

$Eh(Y_i;\theta) = 0$, with Y_i being i.i.d..

[see e.g. Hahn(1996), Hall and Horowitz (1996) for basic bootstrap, Brown and Newey (2002) for resampling in the empirical likelihood function that imposes the moment conditions, rather than the empirical likelihood, to improve bootstrap efficiency, and Donovon and Gonzalves (2017), Assumption 1 for CUGMM].

ii) However, a more limited literature considers the time series framework and the residual (parametric) bootstrap. The approaches can differ with respect to the definition of the residuals. For example, Inoue and Shintani (2006) consider specific moment conditions of the type $E(z_t u_t) = 0$ with $u_t = y_t - \theta_0' x_t$, where (x_t, y_t, z_t) is a strictly stationary observable process. Then, they bootstrap the residuals $\hat{u}_{T,t} = y_t - \hat{\theta}_T' x_t$ that are demeaned and standardized. This approach is convenient when the moment conditions are linear in error u_t. This model includes the causal AR(p) models with i.i.d. errors and where z_t denotes the lagged values of y_t. It is easy to extend this approach to a nonlinear dynamic framework $u_t = g(\tilde{Y}_t, \theta)$ in our notation, with $\tilde{Y}_t = (Y_t, Y_{t-1}, ..., Y_{t-p})$, including the ARCH and ARCH-in-mean models [see Escanciano (2007)].

In our framework, the moment or covariance conditions involve nonlinear transformations of an error u_t and their lagged values. Such a framework is considered in Wan and Davis (2022) with moment (covariance) conditions of the type $E[\exp(ivu_t + iwu_{t+h})]$, where $i = \sqrt{-1}$, and v, w are real numbers (assuming $dim(u_t) = 1$). In Section 5.4, we adopt the type of regularity conditions introduced in Escanciano (2007) and Wan and Davis (2022) for the validity of the residual (parametric) bootstrap. A refinement of the bootstrap approximation is obtained since the test statistic is asymptotically pivotal under the null[10].

5.4 Regularity Conditions for Bootstrapping the Test Statistic

In the literature, an approach with a finite number of conditions, which are nonlinear in u_t has not yet been developed. Inoue and Shintani (2007) consider a finite number of moment conditions that are linear in u. Escanciano (2007) considers an infinite number of moment conditions linear in u, while Wan, Davis (2022) study an inifinite number of nonlinear sine and cosine transformations. The approach proposed in our paper is based on a statistic that is nonlinear in u with a finite number of nonlinear transformations and differs from the literature in this respect.

For ease of exposition, let us consider $H = 1$, $dim(u_t) = 1$, $u_t(\theta_0)$ with a symmetric density f_0 satisfying:

$$\sup_u f_0(u)/\exp(b|u|), \infty, \text{ for } b > 0, \tag{5.2}$$

and a GCov estimator based on a finite number of transformations $a_k(u) = |u|^{p_k} \exp(-t_k|u|), k = 1, ..., K$, where p_k, t_k are fixed. The tail condition (5.2) ensures the uniform integrability of

[10] Cavaliere, Nielsen and Rahbek (2020) develop a bootstrap approach for the OLS estimator of the autoregressive coefficient ρ in a noncausal AR(1) model: $y_t = \rho y_{t+1} + u_t$, where u_t has a stable distribution. It is important to note that the coefficient ρ is not equal to $Cov(y_t, y_{t-1})/Var(y_{t-1})$, because the theoretical moments do not exist. In this respect, the OLS estimator is not a moment (or covariance) estimator.

all moments $Ea_k(u_t(\theta_0))$ by considering transformations that reduce the effect of the tail. We also assume $g(\tilde{y}_t;\theta) = g(y_t, ..., y_{t-p};\theta)$.

We distinguish the following regularity conditions for bootstrap validity:

i) Regularity condition for deriving the asymptotic distribution of the test statistic under the null hypotheses of i.i.d. errors $u_t = u_t(\theta_0)$.

ii) Conditions concerning the third term in the Edgeworth expansion for the refinement of the bootstrap procedure, if the test statistic is asymptotically pivotal under the null.

iii) Additional regularity conditions to ensure the consistency of the bootstrap approach.

i) Regularity condition for deriving the asymptotic distribution of the bootstrap adjusted portmanteau test statistic under $H_{0,i.i.d.}$.
A set of sufficient conditions has been given in Gourieroux, Jasiak (2023), Assumptions A.1, A.2. These include the hypothesis of i.i.d. u_t's, the asymptotic identifiability of parameter θ, and the assumption of invertibility of $\lim_{T\to\infty}[\frac{1}{T}\frac{\partial^2 \xi_T(\theta_0)}{\partial\theta\partial\theta'}]$.

As a consequence of these regularity conditions, we have the asymptotic equivalence of the GCov estimator:

$$\sqrt{T}(\hat{\theta}_T - \theta_0) \equiv \frac{1}{\sqrt{T}}\sum_{t=1}^{T} m_T(\underline{u_t(\theta_0)}, \theta_0) + o_p(1), \tag{5.3}$$

where $\underline{u_t(\theta_0)} = (u_t, u_{t-1}, ...)$, and $m_T(\underline{u_t}, \theta_0)$ is a vector-valued function satisfying the martingale difference sequence condition $E(m_T(\underline{u_t}, \theta_0)|\underline{u_{t-1}}) = 0$, $E||m_T(\underline{u_t}), \theta_0||^2 < \infty$. Then, we have the convergence in the distribution of the empirical process constructed from $\frac{1}{\sqrt{T}}m_T(\underline{u_t(\theta_0)}, \theta_0)$ to a Gaussian process (as a triangular array), and we deduce the asymptotic distribution of the test statistic from the asymptotic expansion of this statistic [see, Gourieroux, Jasiak (2023) for the expansion].

ii) Conditions for refinement

The function m_T in (5.3) is not uniquely defined, since it can always be modified by a term $B(\theta_0)/T$ of order $1/T$. Additional conditions can be introduced, especially on the third-order derivatives of $g(\tilde{y}_t, \theta)$ with respect to θ [see, e.g. Leucht and Neuman (2003)]. Then, we get:

$$\sqrt{T}(\hat{\theta}_T - \theta_0) \equiv \frac{1}{\sqrt{T}}\sum_{t=1}^{T} m_T(\underline{u_t(\theta_0)}, \theta_0) + o_p(1/T). \tag{5.4}$$

In this expansion, the function m_T depends on the function g and the transforms $a_k, k = 1, ..., K$. It has a closed-form expression, even though it is complicated. In fact, we only need the existence of such an expansion, and the convergence in distribution of the empirical process based on $\frac{1}{\sqrt{T}}\sum_{t=1}^{T} m_T(\underline{u_t(\theta_0)}, \theta_0)$ to a Gaussian process. This will imply an asymptotically pivotal test statistic at order 1 under (5.3), at order $1/T$ under (5.4).

iii) Regularity conditions to ensure the consistency of the bootstrap under $H_{0,iid}$.

These conditions have to be introduced for the validity of the bootstrap under the null hypothesis $H_{0,iid}$. Below, we describe sets of sufficient conditions introduced in Wan and Davis (2022) that are specific to this step of bootstrap procedure[11]. They are directly written on functions m_T involved in the expansion (5.3) (resp.(5.4)).

These specific regularity conditions are the assumptions M3, M3', M1' in Wan and Davis (2022). We provide them below for our framework. They are valid for causal models when u_t is a nonlinear innovation of process (y_t)[12]

Assumption M3: $\frac{1}{\sqrt{T}}\sum_{t=1}^{T} |a_k(\hat{u}_{T,t}) - a_k(\hat{u}_{\infty,t})|^m = o_p(1), k = 1, ..., K,\ m = 1, 2,$
where $\hat{u}_{\infty,t}$ denotes the fitted residual based on an infinite sequence of observations.

Assumption M3': For any $\epsilon > 0$,

$P_T[\frac{1}{\sqrt{T}}\sum_{t=1}^{T} |a_k(\hat{u}^s_{T,t}) - a_k(\hat{u}^s_{\infty,t})|^m > \epsilon] \to 0, k = 1, ..., K,\ m = 1, 2,$
when T tends to infinity, where $\hat{u}^s_{\infty,t}$ denotes the bootstrapped residuals based on an infinite sequence of observations.

In comparison with the analogous assumption in Wan and Davis (2022), the condition $u_t(\theta_0) = g(y_t, ..., y_{t-p}; \theta_0)$ is equivalent to $\exp(-t_k u_t(\theta_0)) = \exp(-t_k g(y_t, ..., y_{t-p}; \theta_0))$ for example. Thus, a part of the nonlinearity is solved by changing the representation of the nonlinear autoregressive process.

Assumption M1': For any $\epsilon > 0$ and some $\tau > 0$:

$P_T(|\frac{1}{T}\sum_{t=1}^{T} E_T[m_T'(\underline{\hat{u}^s_{T,t}}, \hat{\theta}_T) m_T(\underline{\hat{u}^s_{T,t}}, \hat{\theta}_T)|\underline{\hat{u}^s_{T,t-1}}] - \tau^2| > \epsilon) \xrightarrow{p} 0,$
when $T \to \infty$ and

$P_T(\frac{1}{T}\sum_{t=1}^{T} E_T[m_T'(\underline{\hat{u}^s_{T,t}}, \hat{\theta}_T) m_T(\underline{\hat{u}^s_{T,t}}, \hat{\theta}_T) 1_{||m_T(\underline{\hat{u}^s_{T,t}}, \hat{\theta}_T)|| > \sqrt{T}\epsilon}|\underline{\hat{u}^s_{T,t-1}}] > \epsilon) \xrightarrow{p} 0,$
when $T \to \infty$, where P_T, E_T are taken with respect to the bootstrap sampling conditional on data, and $\xrightarrow{p}$ is the convergence in probability for the data uncertainty.

Assumption M1' ensures that the martingale difference sequence condition is asymptotically satisfied at order 1 for the bootstrapped residuals[13]

Then, we can apply Theorem 4.2 in Wan and Davis (2022) to obtain the consistency of the bootstrap-adjusted GCov test statistic:

[11]See also Escanciano (2007), Assumption A.5 for a sufficient high-level condition for the validity of the bootstrap, when the moment condition is linear in u_t.

[12]They can also be used for pure noncausal processes, i.e. when u_t is a nonlinear innovation in reversed time. In this case, the innovations as well as their bootstrapped values have to be defined in reversed time.

[13]The rate $\sqrt{T}$ in the second condition in M1' can be modified for refined bootstrap.

$$\sup_z |P_T[\hat{\xi}_T^s < z] - P[\tilde{\xi}_T < z]| \xrightarrow{p} 0,$$

where P_T stands for conditional on data, and $\tilde{\xi}_T$ is the value of ξ_T adjusted for its first-order bias.

Remark 6: In Section 4.1, we noted the difference between the implicit null hypothesis $H_{0,\mathcal{A}}$ and the i.i.d. null hypothesis $H_{0,i.i.d.}$. As such, the test based on a set $\mathcal{A}$ of transformations is not necessarily consistent with respect to the alternative $H_{0,\mathcal{A}} - H_{0,i.i.d.}$. Due to possible conditional heteroscedasticity, the asymptotic distribution of the GCov test statistic under $H_{0,\mathcal{A}} - H_{0,iid}$ can be a mixture of chi-square distributions with the weights depending on the unknown true distribution f_0. Li and Zhang (2022) propose a random weight bootstrap method for the estimation of these weights.

5.5 Bootstrap Analysis of the (Local) Power of Test

The bootstrap can also be used to obtain the approximations of the power and local power of the bootstrap size adjusted overidentification test in the spirit of Davidson and MacKinnon (2006) (see also Escanciano (2007), Section 2.3 for local power). For the power under either fixed or local alternatives, the steps of this approach are as follows:

Step 1: Estimate the model $g(\tilde{y}_t, \theta, \gamma) = u_t,\ t = 1, .., T$ under the alternative to get $\hat{\theta}_{1,T}, \hat{\gamma}_{1,T}$.

Step 2: Find the residuals under the alternative: $\hat{u}_{1,T,t} = g(y_t, \hat{\theta}_{1,T}, \hat{\gamma}_{1,T})$.

Step 3: Get the bootstrapped residuals $\hat{u}^s_{1,b,T,t}$ by drawing in the sample distribution of $\hat{u}_{1,T,t}, t = 1, ..., T$ for $s = 1, ..., S$.

Step 4: Calculate the bootstrapped pseudo-observations $y^s_{1,b,T,t}$ from

$$g(\tilde{y}^s_{1,b,T,t}; \hat{\theta}_{1,T}, \hat{\gamma}_{1,T}) = \hat{u}^s_{1,b,T,t},\ s = 1, ..., S.$$

$$\iff y^s_{1,b,T,t} = c(\underline{\tilde{y}^s_{1,b,T,t-1}}, \hat{u}^s_{1,b,T,t}, \hat{\theta}_{1,T}, \hat{\gamma}_{1,T}),\ s = 1, ..., S.$$

Step 5: Compute from the values $y^s_{1,b,T,t}, t = 1, ..., T$ bootstrapped under the alternative, the estimate $\hat{\theta}^s_{0,b,T}$ of the estimator of θ under the null and the associated test statistic $\hat{\xi}^s_{0,b,T}, s = 1, ..., S$. Then, the empirical distribution of $\hat{\xi}^s_{0,b,T} - \hat{\xi}_{0,T}, s = 1, ..., S$ approximates the distribution of $\hat{\xi}_{0,T}$ under the alternative.

6 Finite Sample Performance of NLSD, GCov Specification and GCov Bootstrap Tests

This section examines the finite sample performance of the proposed test statistics in selected causal-noncausal (vector) autoregressive processes. We perform simulations to study the empirical size and power of a) the NLSD test statistic (2.5) for testing for the absence of (non)linear serial dependence in time series [Section 2.2], b) the GCov specification test statistic (3.3) [Section 3.2], and c) the GCov bootstrap test (4.1) [Section 4].

As the GCov bootstrap test statistic is computed from the residuals of parametric models, Section 5.1 reviews the alternative (Approximate) Maximum Likelihood available for the estimation of causal-noncausal models.

6.1 Parametric Estimation of Causal-Noncausal Models

When a causal-noncausal model is fully parametric, and the errors are assumed to follow a parametric density, the Approximate Maximum Likelihood (ML) estimation can be applied to the noncausal (mixed) processes. The Approximate Maximum Likelihood (AML) estimator of univariate MAR(r,s) processes defined in equation (3.2) and introduced by Lanne, Saikkonen (2011) is:

$$(\hat{\Psi},\, \hat{\Phi},\, \hat{\theta}) = Argmax_{\Psi,\Phi,\theta} \sum_{t=r+1}^{T-s} \ln\, f[\Psi(L^{-1})\Phi(L)y_t; \gamma],$$

where $f[.;\gamma]$ denotes the non-Gaussian probability density function of u_t, such as a t-student density, for example. Davis and Song (2020) discuss the ML estimator for the multivariate causal-noncausal VAR process given in Example 3, Section 3.1, with a parametric error density.

6.2 Simulation Study

We consider the univariate MAR(r,s) processes with i.i.d. errors from non-Gaussian error distributions. We generate univariate processes in samples of size $T = 100, 200, 500$. We consider i.i.d. errors with a uniform distribution $\mathcal{U}_{\ [-1,1]}$, a Laplace distribution with mean zero and variance one, and a t-student (t(5)) distribution with 5 degrees of freedom with mean zero and variance 5/3.

6.2.1 Data Generating Process

In the univariate framework, we apply the simulation method proposed by Gourieroux and Jasiak (2016) to generate the causal-noncausal processes. The MAR(r,s) model is given in equation (3.7) where r is the order of causal polynomial and s is the order of noncausal polynomial. For $r = 0$ and $s = 1$, we generate the MAR(0,1), i.e. the noncausal autoregressive process of order 1:

$$y_t = \psi y_{t+1} + u_t \ \ , |\psi| < 1. \tag{6.1}$$

By setting $r = 1$ and $s = 1$, we generate MAR(1,1) process (3.8). It follows from Lanne, Saikkonen (2011) that it has the following unobserved components $v_{1,t}, v_{2,t}$ defined by:

$$v_{1,t} \equiv (1 - \phi L)y_t \leftrightarrow (1 - \psi L^{-1})v_{1,t} = u_t, \ \ v_{2,t} \equiv (1 - \psi L^{-1})y_t \leftrightarrow (1 - \phi L)v_{2,t} = u_t, \tag{6.2}$$

which can be interpreted as the "causal" and "noncausal" components and used for simulation and bootstrapping. Gourieroux, Jasiak (2016) show that i) $v_{1,t}$ is u-noncausal (i.e. a function of present and future values of u_t) and y-causal (i.e. a function of present and past values of y) and ii) $v_{2,t}$ is u-causal (i.e. a function of present and past values of u_t) and y-noncausal (i.e. a function of present and future values of y). Process y_t has the following deterministic representations based on unobserved components that is used for simulation and bootstrapping of y_t:

$$i)\, y_t = \frac{1}{1 - \phi\psi}(\phi v_{2,t-1} + v_{1,t}), \ \ ii)\, y_t = \frac{1}{1 - \phi\psi}(v_{2,t} + \psi v_{1,t+1}), \tag{6.3}$$

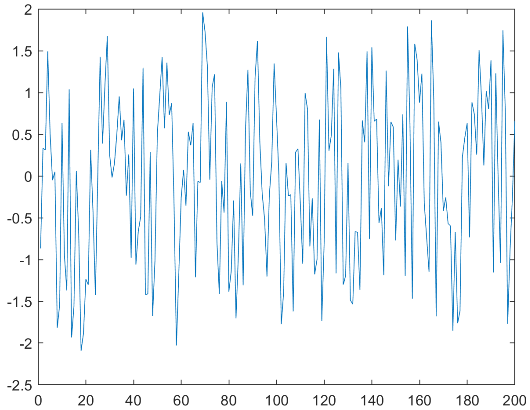
where in i) y_t is a linear function of the first lag of $v_{2,t}$ and of the current value of $v_{1,t}$, and in ii) y_t is a linear function of the current value of $v_{2,t}$ and of the first lag of $v_{1,t}$. The values of coefficients $|\phi| < 1$ and $|\psi| < 1$ are chosen so that the strict stationarity conditions are satisfied.

Figure 1 shows the examples of trajectories of the MAR(0,1) and MAR(1,1) processes simulated with the three error distributions given above.

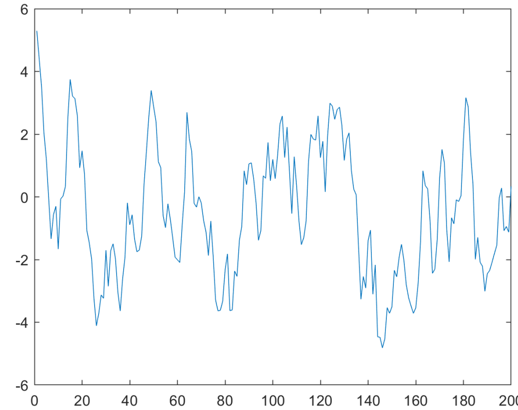
We observe that a large error value creates a spike in the trajectory of the MAR(1,1) with an explosion rate of about $1/\psi$ and a collapse rate of ϕ. In pure processes, we observe a jump if $\psi = 0$ and $\phi > 0$, and an explosive bubble if ψ is small, positive, and $\phi = 0$.

6.2.2 NLSD Test for Time Series

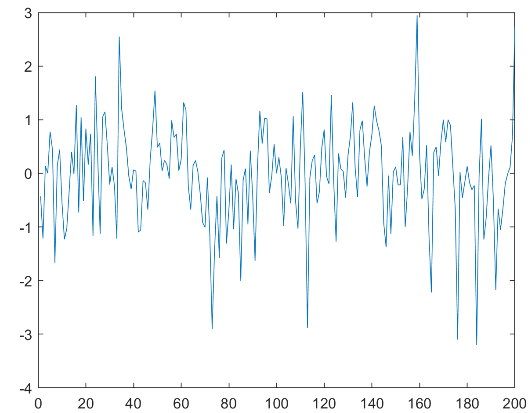
This section examines the size and power of the (non)linear serial dependence (NLSD) test of y_t computed from transformations a of a univariate time series y_t. We consider two



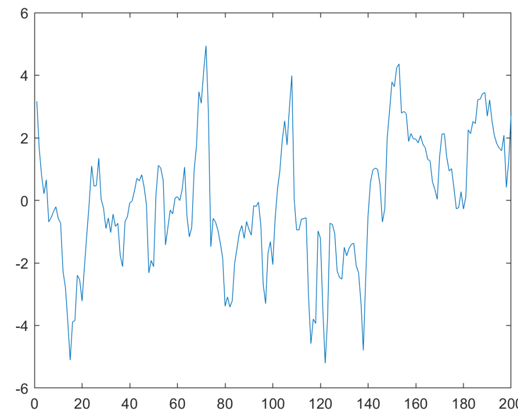
(a) MAR(0,1), $u_t \sim U$, $\psi_1 = 0.2$



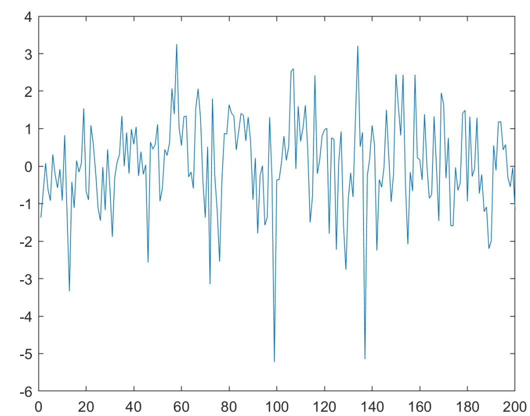
(b) MAR(1,1), $u_t \sim U$, $\phi = 0.2, \psi = 0.8$



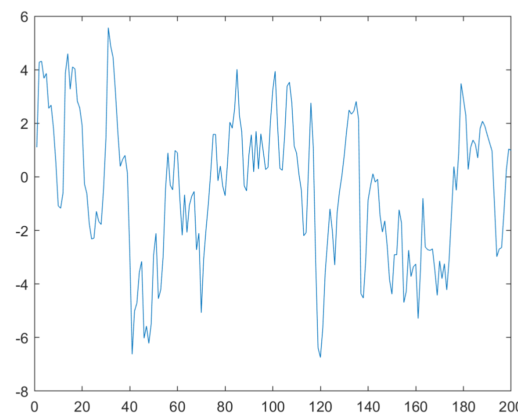
(c) MAR(0,1), $u_t \sim L$, $\psi_1 = 0.2$



(d) MAR(1,1), $u_t \sim L$, $\phi = 0.2, \psi = 0.8$



(e) MAR(0,1), $u_t \sim t(5)$, $\psi_1 = 0.2$



(f) MAR(1,1), $u_t \sim t(5)$, $\phi = 0.2, \psi = 0.8$

Figure 1: Plots of noncausal univariate processes, $T = 200$; L.:Laplace, U.:Uniform, t(5): t-Student with d.f.=5

Table 1: Test of the absence of (non)linear dependence against fixed MAR(0,1) alternative at 5% significance level: size and power

γ	T=100			T=200			T=500		
	Uniform	Laplace	t(5)	Uniform	Laplace	t(5)	Uniform	Laplace	t(5)
0.0	0.0414	0.0484	0.0512	0.0450	0.0502	0.0518	0.0496	0.0540	0.0480
0.1	0.0878	0.0706	0.0684	0.1502	0.1360	0.1220	0.3760	0.3390	0.3572
0.2	0.2682	0.2202	0.2240	0.5618	0.5390	0.5322	0.9586	0.9546	0.9570
0.3	0.6014	0.5742	0.5674	0.9202	0.9266	0.9256	1	0.9998	1
0.4	0.8754	0.8814	0.8754	0.9966	0.9988	0.9976	1	1	1
0.5	0.9796	0.9822	0.9856	1	1	1	1	1	1
0.6	0.998	0.9986	0.9998	1	1	1	1	1	1
0.7	1	1	1	1	1	1	1	1	1
0.8	1	1	1	1	1	1	1	1	1
0.9	1	1	1	1	1	1	1	1	1

The first row ($\gamma = 0$) shows the empirical size of the test and the remaining rows show the size-adjusted power with respect to fixed alternatives.

transformations of the time series: y_t, y_t^2, i.e. $K = 2$, and lag $H = 1$ where the true error distribution f_0 is either Uniform, Laplace, or t(5). and the null hypothesis is the strong white noise hypothesis:

$$H_0 = (\gamma = 0) \equiv (y_t = u_t)$$

The alternative hypothesis is a MAR(0,1) model with coefficients that vary between 0.1 to 0.9:

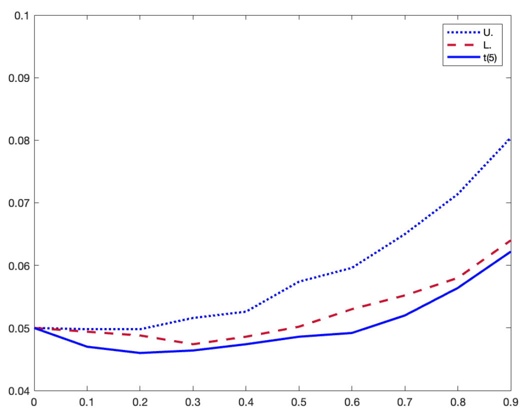
$$H_1 = \{\gamma, f : (1 - \gamma L^{-1})y_t = u_t\}.$$

For each value of γ and each f_0, we simulate the series $y_1^s, ..., y_T^s, s = 1, ..., S$ with S=5000 replications. The nominal size of the test is $\alpha = 0.05$.

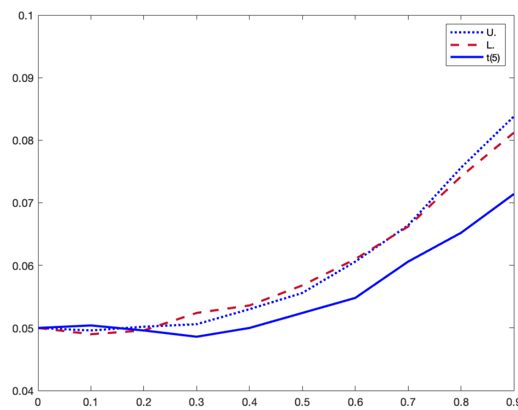
We first consider a fixed alternative, assuming a given density. The first row of Table, 1 with zero values of the autoregressive coefficients provides the empirical size. The remaining rows illustrate the size-adjusted empirical power of the test with respect to fixed alternatives of a $MAR(0, 1)$ process with coefficients that vary between 0.1 and 0.9. The columns of Table 1 pertain to different sample sizes and the Uniform, Laplace, and t(5) error distributions.

The results reported in Table 1 show the good empirical size and power of the test with respect to fixed alternatives, given each density. We observe that higher values of autoregressive coefficients increase the power of the test. When the sample size increases, the power converges to 1 and the size converges to 0.05.

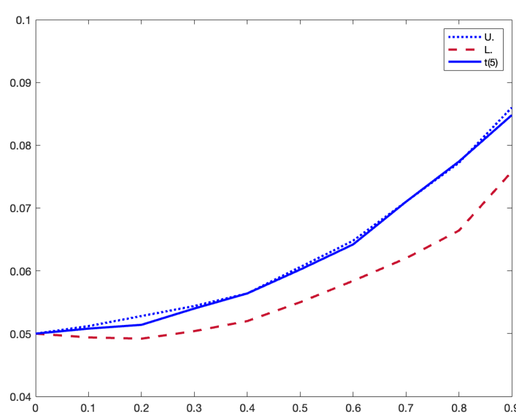
Furthermore, we investigate the power of the test under the local alternatives by generating MAR(0,1) models with coefficients equal to $\frac{\delta}{\sqrt{T}}$, where δ varies between 0 and 0.9. The results are provided in Figure 2. Since we consider the local alternatives, we expect asymptotically the powers to be close to the size for small δ, while for bigger δ, we deviate further away.



(a) $T = 100$



(b) $T = 200$



(c) $T = 500$

Figure 2: Local asymptotic power of the test of the absence of (non)linear dependence.

We see that the test has good local power and the power functions in the neighborhood of the null hypothesis increase fast nonlinearly in δ. The local asymptotic properties appear to depend on the sample size and error distribution. When the sample size increases from 100 to 500, we observe more improvement for the Uniform and t(5)-distributions compared to the Laplace distribution, which has the least heavy tails among these three distributions.

6.2.3 GCov Specification Test for Semi-Parametric Models

This section examines the empirical size and power of the GCov specification test applied to nonlinear transforms of the residuals of a model. We illustrate the bootstrap-adjusted GCov test discussed in Section 5. Next, we examine through simulations the application of the bootstrap test to semi-parametric models estimated by the AML. Under the null hypothesis of "correct specification", i.e. strong white noise errors, the model is a MAR(0,1) with errors $u_t = (1-\psi L^{-1})y_t$. The models are estimated by the GCov estimator with the lag length H=3. We consider K=2 number of transformations with the residuals $\hat{u}_{T,t}$ and squared residuals $\hat{u}^2_{T,t}$ as the non-linear transformations, where $\hat{u}_{T,t} = (1-\hat{\psi}_T L^{-1})y_t$, for the MAR(0,1) process.

Empirical size

To study the size of the test, we compute the nonlinear autocovariances,i.e. the autocovariance matrices of nonlinear transformations of the residuals of the estimated models, denoted by $\Gamma_a(h; ., f)$. The null hypotheses considered are:

$$H_{0,a} = \{\psi, f : \Gamma_{0,a}(h; \psi, f) = 0, \ \forall h = 1, ..., H\},$$

They are tested against the general alternatives

$$H_{1,a} = \{\psi, f : \exists h : \text{such that } \ \Gamma_{0,a}(h; \psi, f) \neq 0, \}.$$

We first consider the GCov specification test of the MAR(0,1) model and generate the noncausal MAR(0,1) processes with autoregressive coefficient values equal to 0.3 and 0.7. All the results are based on S= 5000 replications. The nominal size is $\alpha = 0.05$.

We illustrate the empirical size of the GCov specification test applied to the MAR(0,1) model under the null hypothesis, with the Uniform, Laplace, and t(5) error distributions in first two rows of Table 2 for ψ equal to 0.3 and 0.7. The columns pertain to the different sample sizes and error distributions.

According to these results, the GCov specification test is conservative at $T = 100$ in all cases. When the sample size increases, the empirical size of the GCov test approaches the nominal size for all distributions.

Power of the test

Next, we investigate the performance of the GCov specification test in terms of its empirical power.

Table 2: Empirical size and power of GCov specification test for MAR(0,1) at 5% significance level

S./P.	ϕ	ψ	T=100			T=200			T=500		
			Uniform	Laplace	t(5)	Uniform	Laplace	t(5)	Uniform	Laplace	t(5)
S.	0	0.3	0.0224	0.0454	0.0386	0.0348	0.0566	0.0534	0.0406	0.0544	0.0560
	0	0.7	0.0200	0.0400	0.0338	0.0298	0.0528	0.0468	0.0408	0.0552	0.0528
P.	0.8	0.3	0.1016	0.1724	0.1788	0.3404	0.4468	0.4672	0.9092	0.9282	0.9300
	0.8	0.7	0.9680	0.9882	0.9896	1	1	1	1	1	1

S.: empirical size, P.: empirical power

In the univariate time series, an interesting way to investigate the empirical power of the GCov test is to consider the null hypothesis of MAR(0,1) and deviate from it by adding a causal component and then increasing that causal coefficient. This transforms the MAR(0,1) model under the null hypothesis into a MAR(1,1) model under the alternative.
Therefore, the alternative hypothesis is:

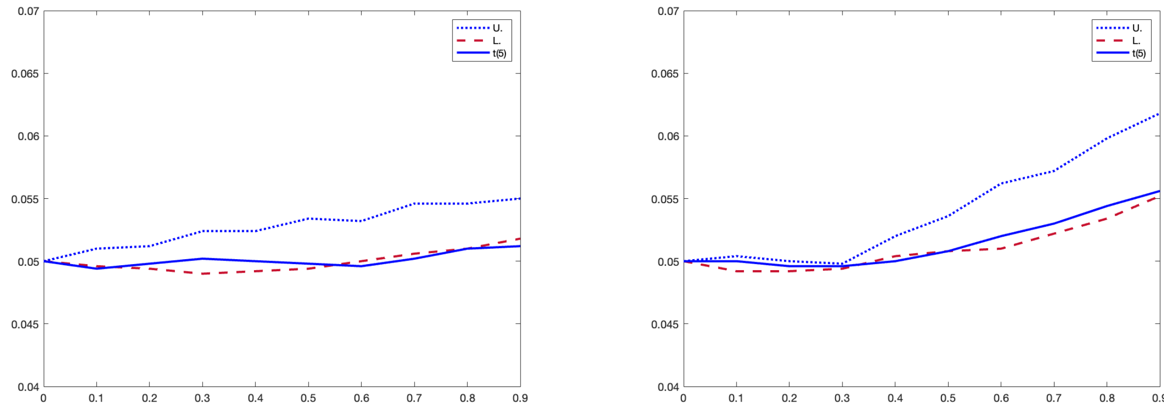
$$H_{1a} = (1 - \phi L)(1 - \psi L^{-1})y_t = u_t,$$

with $\theta = \psi, \gamma = \phi$.

The MAR(0,1) model is estimated by the GCov estimator with the lag length H=3 and the number K=2 of transformations: $\hat{u}_t$ and $\hat{u}_t^2$. We use S=5000 replications and consider the nominal size of 0.05.

In Table 2, we provide the results on the empirical power of the specification tests applied to a noncausal MAR(0,1) model with coefficients ψ equal to either 0.3 or 0.7. In Rows 3 and 4 we consider the additional causal coefficient $\gamma = \phi$ of the MAR(1,1) under the fixed alternative with $\phi = 0.8$ for each given error density and sample size reported in the columns. By comparing the results for $\psi = 0.7$ and $\psi = 0.3$ in Table 2, we can conclude that the empirical power increases in ψ for all sample sizes and distributions. Additional results are given in Table 6, Appendix C, which also shows the effect of the causal persistence coefficient ϕ on the empirical power.

Figure 3 illustrates the local asymptotic power of the GCov specification test computed from the sample of size $T = 500$ and size-adjusted. The three local power functions for MAR(0,1) model with Uniform, Laplace and t(5) distributed errors are plotted against $\delta = 0, 0.1, \ldots, 0.9$, where $\phi_T = \gamma_T = \frac{\delta}{\sqrt{T}}$. Panel (a) displays the results for the MAR(0,1) model with $\psi = 0.3$ and panel (b) shows the results for $\psi = 0.7$. We observe that the test has



(a) MAR(0,1) with $\psi = 0.3$

(b) MAR(0,1) with $\psi = 0.7$

Figure 3: Local asymptotic power of GCov specification test

good local power in each case. The best asymptotic power is displayed in both panels by the Uniform distributed errors with the heaviest tails. The local power functions increase faster in panel (b) for the model with higher noncausal persistence than in panel (a). Overall, the rate of increase of local power functions is lower compared with the local asymptotic power of the (non)linear dependence NLSD test displayed in Figure 2. Hence, larger sample sizes are recommended for the specification test.

6.3 GCov Bootstrap Test

In this section, we illustrate by simulations the performance of the bootstrap GCov test. Recall that the GCov bootstrap test can be applied to test the specification of a model estimated by an estimator different from the GCov. To give an understanding of the performance of the GCov bootstrap test, in Figure 4 we show how the distribution of the bootstrap test statistic converges to the distribution of ξ_s when the sample size increases from 100 to 500.

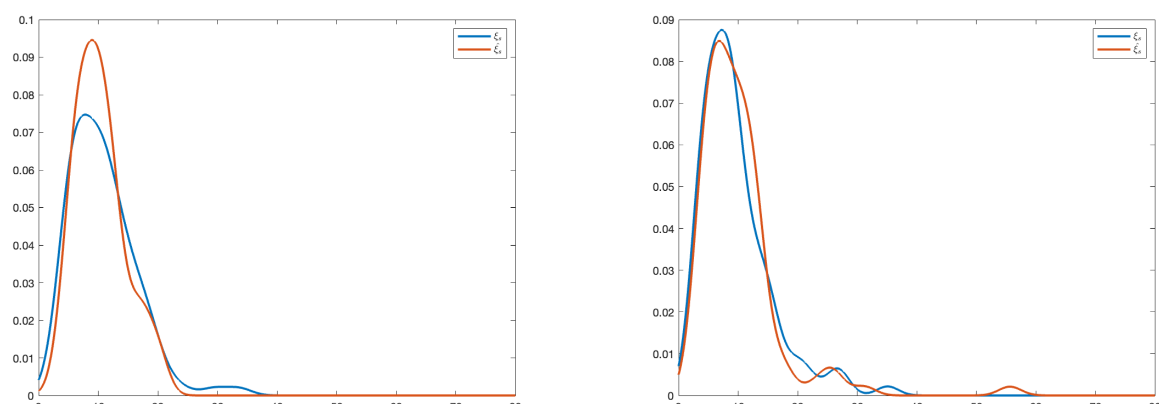
Like in the previous Section, under the null hypothesis, the model is a MAR(0,1), and under the fixed alternative, it is a MAR(1,1), allowing us to study the empirical power. We consider the same series as in Table 2 to allow the comparison of the performance of the asymptotic GCov test and the bootstrap test with an AML estimator. The model is not estimated by the GCov, but instead by the Approximate Maximum Likelihood (AML) estimator described in Section 4.1. We consider the AML estimators based on a t-student log-likelihood function with an estimated degree of freedom fitted to the models with the Uniform, Laplace and t(5) distributed errors. Hence, there is misspecification for the models with Uniform and Laplace error distributions and the Quasi-AML (QAML) estimators are obtained. The

Table 3: Empirical size and power of bootstrap-based specification test for MAR(0,1) at 5% significance level

S./P.	ϕ	ψ	T=100			T=200			T=500		
			Uniform	Laplace	t(5)	Uniform	Laplace	t(5)	Uniform	Laplace	t(5)
S.	0	0.3	0.051	0.069	0.062	0.059	0.056	0.046	0.063	0.064	0.045
	0	0.7	0.055	0.066	0.062	0.064	0.052	0.046	0.066	0.062	0.049
P.	0.8	0.3	0.216	0.395	0.412	0.497	0.652	0.665	0.981	0.975	0.986
	0.8	0.7	0.976	0.987	0.992	1	1	1	1	1	1

S.: empirical size, P.: empirical power

GCov bootstrap test statistic is computed with H=3 and K=2. The experiment is replicated 1000 times. We employ simple bootstrap methods with 100 replications. Table 3 presents the empirical size and power of the GCov bootstrap test. We find that the empirical size of this test converges asymptotically to the nominal level, and it has high power against fixed alternatives. We can argue that the bootstrap-based test has good empirical size and power in small sample despite the misspecification of the models with Uniform and Laplace distributed errors. By comparing the results of Tables 2 and 3, we observe that the bootstrap test provides a close to nominal size in small samples.



(a) $T = 100$ (b) $T = 500$

Figure 4: Comparison of the distributions of ξ_s and $\hat{\xi}_s$ for MAR(0,1) with t(5) error distribution and $\psi = 0.7$

7 Empirical Application

In this section, we apply the GCov specification test to a univariate causal-noncausal model fitted to the series of aluminum prices in U.S. Dollars per metric ton. This approach is motivated by the presence of spikes and bubbles in the aluminum prices, and the recent literature on causal-noncausal modeling of commodity prices [Hecq, Lieb, and Telg (2016), Fries, Zakoian (2019), Gourieroux, Jasiak (2022)]. First, we apply the (non)linear serial dependence NLSD test to the data and next use the GCov specification test to examine the goodness of fit of the causal-noncausal processes fitted to the data.

Our sample consists of $T = 228$ monthly average prices recorded between January 2005 and October 2024 and referred to as the Global price of Aluminum [14]. We detrend the series of prices by regressing it on time (polynomial of degree one). The detrended prices are plotted in Figure 5a, where we observe multiple spikes and a sudden drop in aluminum prices during the 2008 recession when the commodity prices fell due to weak demand. Moreover, in 2020, we see a spike in the price of aluminum, which is due to the weak supply of commodities at the beginning of the Covid period.

To ensure the identification of the causal and noncausal dynamics, we test the data for normality by applying the Kolmogorov-Smirnov normality test. The test statistic of 0.50 exceeds the critical value of 0.08. Hence, the null hypothesis of the normality of aluminum price distribution is rejected. Figure 6 (a) in Appendix C provides the sample density plot of demeaned aluminum price and compares it to the Gaussian density. We confirm that the aluminum prices are non-Gaussian.

We test for nonlinear serial dependence in the aluminum prices using the NLSD test introduced in Section 2. We compute the test statistic from the series using H=9 and K=2. The value of the NLDS test is 1675.4 and exceeds the critical value of 50.99, showing (non)linear serial dependence in the data. This finding is confirmed by the ACF of the series and their squares in Figures 7a and 7b in Appendix C.

Next, we use the semi-parametric approach without any distributional assumptions on the errors and explore several specifications of the causal-noncausal MAR(r,s) models for varying causal and noncausal orders r and s. In Table 4 we report the GCov estimated parameters with H=9 and K=2, where we use the residuals and the logarithm of the absolute value of residuals power two.

We apply the GCov specification test to each model, i.e. the test of the null hypothesis of

[14] International Monetary Fund, Global price of Aluminum [PALUMUSDM], retrieved from FRED, Federal Reserve Bank of St. Louis; https://fred.stlouisfed.org/series/PALUMUSDM, December 20, 2024.

strong white noise errors to assess its fit. These statistics and the associated critical values are given in columns 2 and 3 of Table 4. Under the strict stationarity assumption, all models have autoregressive polynomials with roots outside the unit circle. The estimated roots of $\hat{\Phi}(L)$ and $\hat{\Psi}(L^{-1})$ polynomials are given in the last column of Table 4. We find that for the MAR(1,1) model, the GCov specification test does not reject the null hypothesis of strong white noise residuals, indicating the absence of (non)linear serial dependence in the residuals. Figure 5b shows the in-sample fitted values of the MAR(1,1) model. Figure 6 (b) in Appendix C, illustrates the non-Gaussian distribution of the residuals. The ACF of these residuals and their squares given in Figures 7c and 7d of Appendix C are not statistically significant, which confirms the results of the GCov specification test. Furthermore, we plot the fitted causal and noncausal components of MAR(1,1), i.e. $\hat{v}_{1,t} = (1 - \hat{\psi}L^{-1})y_t$ and $\hat{v}_{2,t} = (1 - \hat{\phi}L)y_t$ in Figure 8 of Appendix C. The components $v_{1,t}, v_{2,t}$ are discussed in Section 6.2.1 and defined in equation (6.2), with $V_{1,t}$ capturing the locally explosive patterns.

Table 4: Estimated parameters of selected causal-noncausal models, GCov specification test with χ^2 critical values at 5% significance level, and roots of $\hat{\Phi}(L^{-1})$ and $\hat{\Psi}(L)$

	ϕ_1	ψ_1	ψ_2	test statistic	$\chi^2_{0.95}$	L_1^{ϕ}	L_1^{ψ}	L_2^{ψ}
MAR(0,1)		0.93*		57.53	49.80	1.07		
MAR(1,1)	**0.41***	**0.87***		**22.32**	**48.60**	**2.43**	**1.14**	
MAR(1,2)	0.62*	0.76*	0.02	22.60	47.4	1.60	1.27	-36.93

* indicates statistical significance at 5%

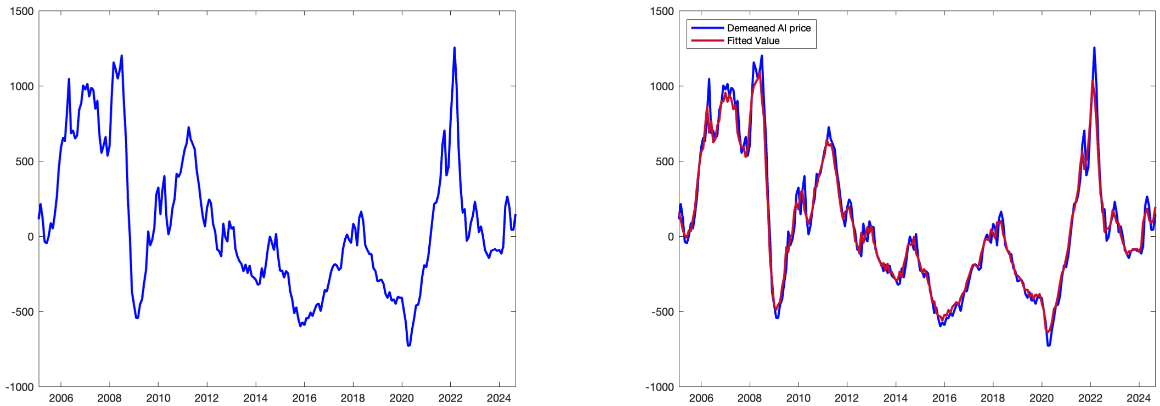
In addition, we estimate the MAR(1,1) model using the AML estimator with the log-likelihood based on the fitted t-student error distribution with 3.9 degrees of freedom[15] to illustrate the GCov bootstrap test. Table 5 displays the estimated causal and noncausal coefficients and GCov bootstrap test with the null hypothesis of i.i.d residuals.

Table 5: Estimated parameters of MAR(1,1) model by AML, GCov bootstrap test with critical values at 5% significance level, and Ljung-Box test of residuals and residuals square

	ϕ	ψ	bootstrap test	CV	LB($\hat{\epsilon}_t$)	$\chi^2_{0.95}(20)$	LB($\hat{\epsilon}_t^2$)	$\chi^2_{0.95}(20)$
MAR(1,1)	0.91*	0.36*	97.14	70.41	14.68	31.41	68.85	31.41

* indicates statistical significance at 5%

[15]The degrees of freedom are an additional AML parameter estimated in this model.



(a) Detrended Aluminum price (b) MAR(1,1) fitted values

Figure 5: Detrended Aluminum price, MAR(1,1) fitted values

By comparing Tables 4 and 5, we find that the AML estimate of the causal coefficient of the MAR(1,1) process is closer to the unit root. Based on the GCov bootstrap test results, we reject the null hypothesis of i.i.d residuals in one step in the AML estimated model. However, the Ljung-Box test does not reject the absence of dependence in the residuals at the level of 5% and can only detect the existence of dependence in the squared residuals. The AML results could suffer from misspecification of the parametric likelihood function, and we see from the portmanteau test results in Tables 4 and 5, that the GCov estimated model has a satisfactory fit while the model based on the AML does not. Moreover, the GCov bootstrap test is advantageous, compared to the Ljung-Box test, since it rejects the null of i.i.d residuals in one step.

8 Conclusion

This paper considers nonlinear serial dependence tests in non-Gaussian time series and specification testing in models with non-Gaussian i.i.d. errors. We examined analytically and through simulations the finite sample properties of the semi-parametric Gcov specification test under the local hypotheses to provide convincing empirical evidence of its potential as a widely applicable diagnostic tool for testing the goodness of fit of semi-parametric dynamic models with i.i.d. non-Gaussian errors.

We introduced a new tests of the null hypothesis of the absence of (non)linear dependence in time series, called the NLSD test. We also introduced a new GCov bootstrap specification test applicable to dynamic non-Gaussian models estimated by a method other than the GCov, such as the semi-parametric GMM or parametric ML-type estimators.

We explored the finite sample performance of these new tests in simulations and described analytically the asymptotic distributions of the test statistics under the local alternatives. The local alternatives were used to focus on specification errors due to the parameters rather than the marginal error density, for example, in the semi-parametric model of interest. For illustration, we applied the NLSD test to the aluminum prices. Next, we used the GCov specification test to select the optimal fit of a causal-noncausal MAR model of aluminum prices.

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REFERENCES

Inoue, A.. and M., Shintani (2006): Bootstrapping GMM Estimators for Time Series", Journal of Econometrics, 133, 531-555.

Anderson, T. (1999): "Asymptotic Theory for Canonical Correlation Analysis", Journal of Multivariate Analysis, 70, 1-29.

Anderson, T. (2002): "Canonical Correlation Analysis and Reduced Rank Regression in Autoregressive Models", Annals of Statistics, 30, 1134-1154.

Andrews, D. (1988): "Laws of Large Numbers for Dependent Non-Identically Distributed Random Variables", Econometric Theory, 4, 458-467.

Andrews, D. (1992): "Generic Uniform Convergence", Econometric Theory, 8, 241-257.

Bartlett, M. (1947): "The General Canonical Correlation Distribution", Annals of Mathematical Statistics, 18, 1-17.

Bickel , P. (1982): "On Adaptive Estimation", Annals of Statistics, 10, 647-671.

Bierens, H.J. (1990): "A Consistent Conditional Moment Test of Functional Form", Econometrica, Vol 58, 1443-1458.

Box, G. and D. Pierce (1970): "Distribution of Residual Autocorrelations in Autoregressive-Integrated Moving Average Time Series Models", JASA, 65, 1509-1526.

Brown, B. (1971): "Martingale Central Limit Theorem", Annals of Mathematical Statistics, 42, 59-66

Brown, B., and W., Newey (2002): "Generalized Method of Moments, Efficient Bootstrapping and Improved Inference", Journal of Business and Economic Statistics, 20, 507-513. Bryan, K. (2006) "Elementary Inversion of the Laplace Transform", Research Paper, Department of Mathematics, Rose-Hulman Institute of Technology.

Boudjellaba, H., Dufour, J.M. and R. Roy (1994): "Simplified Conditions for Noncausality Between Vectors in Multivariate ARMA Models", Journal of Econometrics, 63, 271-287.

Cavaliere, G., Nielsen, H, and A., Rahbek (2020): "Bootstrapping Noncausal Autoregressions with Applications to Explosive Bubble Modelling", Journal of Business and Economic Statistics, 38, 55-67.

Chan, K., Ho, L., Tong, H., (2006): "A Note on Time-Reversibility of Multivariate Linear Processes", Biometrika 93, 221-227.

Chu, B. (2023): "A Distance-Based Test of Independence Between Two Multivariate Time Series", Journal of Multivariate Analysis, 195, 105-151.

Chitturi, R. (1974): "Distribution of Residual Autocorrelations in Multiple Autoregressive Schemes", JASA, 69, 928-934.

Chitturi, R. (1976): "Distribution of Multivariate White Noise Autocorrelations", JASA, 71, 223-226.

Cubbada, G. and A. Hecq, (2011): "Testing for Common Autocorrelation in Data-Rich Environments", Journal of Forecasting, 30, 325-335.

Davidson, R., and J., MacKinnon (2006): "The Power of Bootstap and Asymptotic Test", Journal of Econometrics, 133, 421-441.

Davis, R. and L. Song (2020): "Noncausal Vector AR Processes with Application to Economic Time Series", Journal of Econometrics, 216, 246-267.

Davis, R and P. Wan (2022): "Goodness of Fit Testing for Time Series Models via Distance Covariance", Journal of Econometrics, 227, 4-24.

Davis, R and Wu (1997): "Bootstrapping M-Estimates in Regressions and Autoregressions with Infinite Variances, Statistica Sinica, 7, 1135-1156.

De Jong, R. (1995): "Laws of Large Numbers for Dependent Heterogeneous Processes", Econometric Theory, 11, 347-358.

De Jong, R. (1998): "Weak Laws of Large Numbers for Dependent Random Variables", Annales d'Economie et de Statistique, 51, 210-225.

De Gooijer, J.G. (2023): "On Portmanteau-Type Tests for Nonlinear Multivariate Time Series", Journal of Multivariate Analysis, 195, 105-157.

Dovonon, P., and S., Goncalves (2017): "Bootstrapping the GMM Overidentification Test Under First-Order Underidentification," Journal of Econometrics, 201, 43-71.

Dovonon, P., Hall, A., and F. Kleibergen (2020): "Inference in Second -Order Identified Models", Journal of Econometrics, 218, 346-372

Dovonon, P. and N. Gospodinov (2024): "Specification Testing for Conditional Moment Restrictions Under Local Identification Failure", Quantitative Economics, forthcoming

Duchesne, P. and R. Roy (2003): "Robust Tests for Independence of Two Time Series", Statistica Sinica, 13, 827-852.

Duchesne, P. and R. Roy (2004): "On Consistent Testing for Serial Correlation of Unknown Form in Vector Time Series Models" Journal of Multivariate Analysis, 89, 148-180.

Dvoretski, A. (1970): "Asymptotic Normality for Sums of Dependent Random Variables", Proc. of the Sixth Berkeley Symposium on Math. Stat. and Prob., Vol II, 513-535.

El Himdi, H. and R. Roy (1997): "Tests for Noncorrelation of Two Multivariate ARMA Time Series", Canadian Journal of Statistics, 25, 233-256.

Engle, R., Lilien, D. and R. Robbins (1987): "Estimating Time Varying Risk Premia in the Term Structure: The ARCH-M Models", Econometrica, 55, 391-407.

Escanciano, M. (2007): "Model Checks Using Residual Marked Empirical Processes", Statistica Sinica, 17, 115-138

Feller, W. (1971): "Probability Theory and Its Applications", Vol 3, 2nd Ed, Wiley, Chapter 13: "Laplace Transforms, Tauberian Theorems, Resolvents".

Fokianos, K. and M. Pitsillou (2018): "Testing Independence for Multivariate Time Series via the Autodistance Correlation Matrix", Biometrika, 105, 337-352.

Francq, C., Roy, R. and J.M. Zakoian (2005): "Diagnostic Checking in ARMA Models with Uncorrelated Errors", JASA, 100, 532-544.

Francq, C., and J.M. Zakoian (2005): " A Central Limit Theorem for Mixing Triangular Arrays of Variables Whose Dependence is Allowed to Grow with the Sample Size", Econometric Theory, 21, 1165-1171.

Fries, S., and Zakoian, J. M. (2019):" Mixed causal-noncausal ar processes and the modelling of explosive bubbles", Econometric Theory, 35(6), 1234-1270.

Geweke, J. (1981): "The Approximate Slopes of Econometric Tests", Econometrica, 49, 1427-1442.

Gourieroux, C. and J. Jasiak (2016): "Filtering, Prediction and Simulation Methods for Noncausal Processes", Journal of Time Series Analysis, Vol 37, 405-430

Gourieroux, C. and J. Jasiak (2017): "Noncausal Vector Autoregressive Process: Representation, Identification and Semi-Parametric Estimation", Journal of Econometrics, 200, 118-134.

Gourieroux, C. and J. Jasiak (2018): "Misspecification of Noncausal Order in Autoregressive Processes", Journal of Econometrics, 205(1), 226-248

Gourieroux, C. and J. Jasiak (2019): "Robust Analysis of the Martingale Hypothesis", Econometrics and Statistics, Vol 9, 17-41

Gourieroux, C. and J. Jasiak (2023): "Generalized Covariance Estimator", Journal of Business and Economic Statistics, 41, 1315-1327.

Gourieroux, C. and A. Monfort (1995): " Statistics and Econometric Models", Vol 2, Cambridge Univ. Press.

Gourieroux, C., A. Monfort, and J.P. Renne (2017), "Statistical Inference for Independent Component Analysis: Application to Structural VAR Models", Journal of Econometrics 196, 111-126.

Gourieroux, C., A. Monfort, and J.P. Renne (2018), "Identification and Estimation in Non-Fundamental Structural Models", Review of Economic Studies, 2020, 87 (4), pp.1915-1953.

Hahn, J. (1996): "A Note on Bootstrapping Generalized Method of Moments Estimators", Econometric Theory, 12, 187-197.

Hall, P., and J., Horowitz (1996): "Bootstrap Critical Values for Tests Based on Generalized Method of Moments Estimators", Econometrica, 64, 831-916.

Han, C. and P.C.B. Phillips (2006): "GMM with Many Moment Conditions", Econometrics, 74, 147-192.

Hannan, J. (1967): "Canonical Correlation and Multiple Equation Systems in Economics", Econometrica, 35, 123-138.

Hannan, J. (1976): "The Asymptotic Distribution of Serial Covariances", Annals of Statistics, 4, 396-399.

Haugh, L. (1976): "Checking the Independence of Two Covariance-Stationary Time Series: A Univariate Residual Cross-Correlation Approach", JASA, 71, 378-385.

Hecq, A., Issler, J. V., and Telg, S. (2020): "Mixed Causal-Noncausal Autoregressions with Exogenous Regressors", Journal of Applied Econometrics, 35(3), 328-343.

Hecq, A., Lieb, L., and Telg, S. (2016): "Identification of Mixed Causal-Noncausal Models in Finite Samples", Annals of Economics and Statistics, (123/124), 307-331.

Hecq, A., and Sun, L. (2020): "Selecting Between Causal and Noncausal Models with Quantile Autoregressions", Studies in Nonlinear Dynamics and Econometrics, 25(5), 393-416.

Hecq, A. and Voisin, E. (2021): "Forecasting Bubbles with Mixed Causal-Noncausal Autoregressive Models", Econometrics and Statistics, 20, 29-45.

Hosking, J. (1980): "The Multivariate Portmanteau Statistic", JASA, 75, 602-608.

Hosking, J. (1981)a: "Equivalent Forms of the Multivariate Portmanteau Statistic", JRSS B, 43, 261-262.

Hosking, J. (1981)b: "Lagrange Multiplier Tests of Multivariate Time Series Models", JRSS B, 43, 219-230.

Hotelling, H. (1936) : "Relation Between Two Sets of Variants", Biometrika, 28, 321-377.

Jennrich, R. (1969): "Asymptotic Properties of Nonlinear Least Squares", Annals of Mathematical Statistics, 40, 633-643.

Keweloh, S.A. (2020): "A Generalized Method of Moments Estimator for Structural Autoregressions Based on Higher Moments", Journal of Business & Economic Statistics, 39(3), 1-29.

Koenker, R. and J.A.F Machado (1999) "GMM Inference when the Number of Moment Conditions is Large", Journal of Econometrics, 93, 327-344

Kundu, S., Majumdar, S., and K. Mukherjee (2000): "Central Limit Theorems Revisited", Statistics and Probability Letters, 47, 265-275.

Lanne, M., and J. Luoto (2019), "GMM Estimation of Non-Gaussian Struc- tural Vector Autoregression", Journal of Business & Economic Statistics, 69-81

Lanne, M., M. Meitz, and P. Saikkonen (2017), "Identification and Estimation of Non-Gaussian Structural Vector Autoregressions", Journal of Econometrics 196, pp.288-304.

Lanne, M., and P. Saikkonen, (2011): "Noncausal Autoregressions for Economic Time Series", Journal of Time Series Econometrics, 3(3).

Lanne, M., and P. Saikkonen, (2013): "Noncausal Vector Autoregression", Econometric Theory, 29(3), 447-481.

Leucht, A., and M., Neumann (2003): " Consistency of General Bootstrap Methods for degenerate U-Type and V-Type Statistics", Journal of Multivariate Analysis, 100, 1622-1633. Li, W. and A. McLeod (1981): "Distribution of the Residual Autocorrelations in Multivariate ARMA Time Series Models", JRSS B, 43, 231-233.

Li, M. and Y. Zhang (2022): "Bootstrapping Multivariate Portmanteau Tests for Vector Autoregressive Models with Weak Assumptions on Errors", Computational Statistics and Data Analysis, 165, 107321.

Ling, S. (2007): A Double AR(p) Model: Structure and Estimation, Statistica Sinica, 17,161-175.

Mahdi, E., and A. McLeod (2012): "Improved Multivariate Portmanteau Test", Journal of Time Series Analysis, 33(2), 211-222.

Newey, W. (1988): "Adaptive Estimation of Regression Models Via Moment Restrictions", Journal of Econometrics, 38, 301-334.

Post, E. (1930): Generalized Differentiation", Transactions of the American Mathematical Society, 32, 723-781.

Robinson, P. (1973): " Generalized Canonical Analysis for Time Series", Journal of Multivariate Analysis, 3, 141-160.

Royden, H.L., and P.M. Fitzpatrick (2010): "Real Analysis", 4th ed.,Prentice Hall

Swensen, A,R. (2022): "On Causal and Non-Causal Cointegrated Vector Autoregressive Time Series", Journal of Time Series Analysis, 43(2), 178-196.

Szroeter, J. (1983): "Generalized Wald Methods for Testing Nonlinear Implicit and Overidentifying Restrictions", Econometrica, 51, 335-353.

Velasco, C. and I. Lobato (2018): "Frequency Domain Minimum Distance Inference for Possibly Noninvertible and Noncausal ARMA Models", Annals of Statistics, 46, 555-579.

Wan, P., and Davis, R. A. (2022): "Goodness-of-Fit Testing for Time Series Models via Distance Covariance", Journal of Econometrics, 227(1), 4-24.

Wooldridge, J. and H. White (1988): "Some Invariance Principles and Central Limit Theorems for Dependent Heterogenous Processes", Econometric Theory, 4, 210- 230.

Appendix A

The following notation is used:

n - dimensions of Y_t

u_t is of dimension $J = dim(g)$

K - dimension of transformations a

$a(u_t) = g_a$ is of dimension K

Γ or (Γ^a) is of dimension $K \times K$ (except for Section 2.1.2, where Γ is of dimension n)

$dim(\theta)$ - dimension of θ

$dim(\gamma) = 1$, γ is a scalar

Id is the Identity matrix of dimensions n and Km such that $n = K$.

Asymptotic Behavior of the Portmanteau Statistic Under the Independence Hypothesis

This Appendix reviews the results, which already exist in the literature and are used in the proofs of new results in Appendix B.

A.1 Asymptotic Behavior of Sample Autoregressive Coefficients

Suppose that process (Y_t) is strictly stationary and follows a VAR(1) model:

$$Y_t = \alpha + BY_{t-1} + u_t, \tag{A.1}$$

where u_t is a square integrable strong white noise, $E(u_t) = 0$, $V(u_t) = \Sigma$, wher Σ is invertible and the coefficient matrix B has no eigenvalues of modulus 1. This VAR model is a SUR model with identical regressors $X_t = Y_{t-1}$ in all equations. In this case, the OLS estimators applied equation by equation are equal to the GLS estimator of B [16]. The estimator $\hat{B}' = \hat{\Gamma}(0)^{-1}\hat{\Gamma}(1)'$ is asymptotically normally distributed:

$$\sqrt{T}[vec(\hat{B}') - vecB'] \approx N[0, \Sigma \otimes \Gamma(0)^{-1}].$$

Under the null hypothesis: $H_0 = (\Gamma(1) = 0) = (B = 0)$, we have $\Sigma = \Gamma(0)$ and

$$\sqrt{T}vec(B') \sim N(0, \Gamma(0) \otimes [\Gamma(0)^{-1}]).$$

where the $\otimes$ denotes the Kronecker product [see Chitturi (1974), eq. (1.13)].

A.2 Portmanteau Statistic as a Lagrange Multiplier test

[16] In this Appendix the index T of the estimators is omitted to simplify the notation.

It can be shown that the Lagrange Multiplier test statistic for testing $H_0 = (\Gamma(1) = 0) = (B = 0)$ is:

$$\hat{\xi}_T(1) = T\ Tr[\hat{\Gamma}(1)'\hat{\Gamma}(0)^{-1}\hat{\Gamma}(1)\hat{\Gamma}(0)^{-1}] = T\ Tr\hat{R}^2(1). \tag{A.2}$$

[See, e.g. Gourieroux, Jasiak (2023), Supplementary Material].

A.3 Asymptotic Behavior of Sample Autocovariance

The asymptotic distribution of $\sqrt{T}vec[\hat{\Gamma}(1)' - \Gamma(1)']$ for model (A.1) is given in Gourieroux, Jasiak (2023), Supplementary Material [see also Chitturi (1976), Hannan (1976)]. We have

$$\sqrt{T}[\hat{\Gamma}(1)' - \Gamma(1)'] = \hat{\Gamma}(0)\sqrt{T}[\hat{B}' - B'] = \Gamma(0)\sqrt{T}[\hat{B}' - B'] + o_p(1),$$

and

$$vec[\sqrt{T}[\hat{\Gamma}(1)' - \Gamma(1)']] = vec[\Gamma(0)\sqrt{T}[\hat{B}' - B']] = [Id \otimes \Gamma(0)]vec(\sqrt{T}[\hat{B}' - B']) + o_p(1).$$

Under the null hypothesis $H_0 := (\Gamma(1) = 0) = (B = 0)$ of independently and identically distributed (i.i.d.) process (Y_t) with finite fourth order moment:

$$\begin{aligned} vec[\sqrt{T}[\hat{\Gamma}(1)' - \Gamma(1)']] &\sim N[0, [Id \otimes \Gamma(0)][\Gamma(0) \otimes \Gamma(0)^{-1}][Id \otimes \Gamma(0)]] \\ &= N[0, \Gamma(0) \otimes \Gamma(0)]. \end{aligned}$$

It follows that under this null hypothesis, the statistic (A.2) follows asymptotically a chi-square distribution $\chi^2(K^2)$, where $K = n$ is the dimension of (Y_t).

A.4 Statistic Based on Several Autocovariances

The interpretation as a SUR regression can be extended to any lag H. Then, under the stationarity assumption, the VAR model becomes:

$$Y_t = \alpha + B_1 Y_{t-1} + \cdots + B_H Y_{t-H} + u_t, \tag{A.3}$$

where (u_t) is a square integrable strong white noise and the companion matrix of autregressive coefficients has no eigenvalues of modulus 1. Under the null hypothesis of the independence of Y_t, or equivalently under $H_0 = \{B_1 = \cdots = B_H = 0\}$, the explanatory variables are orthogonal, and the OLS estimators of $B_1, ..., B_H$ are such that $\hat{B}_h$ coincides with the OLS estimator in the simple SUR model $Y_t = \alpha_h + B_h Y_{t-h} + v_t$. It follows that, under this null hypothesis, the estimators $\sqrt{T}\hat{B}_h,\ h=1,...,H$ are independent, normally distributed with the same distribution $N(0, \Gamma(0) \otimes \Gamma(0))$. Then, the test statistics:

$$\hat{\xi}_T(H) \approx T \sum_{h=1}^{H} vec[\sqrt{T}\hat{\Gamma}(h)']'[\hat{\Gamma}_0(0)^{-1} \otimes \hat{\Gamma}_0(0)^{-1}]vec[\sqrt{T}\hat{\Gamma}(h)'] \quad \text{(A.4)}$$

follows asymptotically the chi-square distribution $\chi^2(K^2H)$, where $K=n$ is the dimension of (Y_t).

Appendix B
Asymptotic Distribution in the Semi-Parametric Framework

This Appendix provides the regularity conditions and proofs of Propositions 1, 2, B1 and B2.

B.1 The Law of Large Numbers (LLN) for Triangular Arrays

As pointed out in Section 3.3.2, the proof of the consistency of estimated autocovariances and of the GCov estimator under the local alternatives is similar to the proof under the null hypothesis of independence. The only difference is in the use of the LLN for empirical autocovariances of a triangular array of observations, uniform in θ.

Below, we provide a sufficient set of regularity conditions.

Regularity Conditions for LLN uniform in θ.

1. Conditions on the true nonlinear dynamics

i) The observations satisfy the model:

$$g^*(\tilde{Y}_{T,t}; \theta_T, \gamma_T) = u_t, \quad \text{(B.1)}$$

where the u_t's are i.i.d. with pdf f_0.

ii) The function g^* is invertible with respect to $Y_{T,t}$; then we can write:

$$Y_{T,t} = h(u_t, Y_{T,t-1}, ..., Y_{T,t-p}; \theta_T, \gamma_T). \quad \text{(B.2)}$$

iii) For each given T, $(Y_{T,t})$ with a varying t, is a strictly stationary and ergodic solution of the autoregressive equation (B.2).

2. Conditions on the parameters

Suppose that the parameter space is $\Theta \times C$, where $\theta \in \Theta \subset \mathbf{R}^{dim(\theta)}$ and $\gamma \in C \subset \mathbf{R}$. We assume that:

i) Θ and C are compact sets with non-empty interiors.

ii) θ_0 is in the interior of Θ and 0 is in the interior of C.

iii) $\theta_T = \theta_0 + \mu/\sqrt{T},\ \gamma_T = \nu/\sqrt{T}$.

In particular, for T sufficiently large, θ_T, γ_T are in the interior of Θ and C, respectively.

3. Regularity conditions on function g^*

i) The functions $g_k^*(y;\theta,\gamma),\ k = 1,...,K$ are continuously differentiable on the interior $\Theta \times C$.

ii) Let us define: $G_k^*(\tilde{y}) = Max_{(\theta,\gamma)\in\Theta\times C}[g_k^*(\tilde{y},\theta,\gamma)]^2$. We assume $E_0 G_k^*(\tilde{Y}) < \infty,\ k = 1,...,K$ where E_0 denotes the expectation computed for the process $(\tilde{Y}_t)$ associated with the "asymptotic" parameter values $(\theta_0, 0)$.

iii) Let us denote by $\mathcal{B}(\tilde{y})$ a uniform Lipschitz coefficient for functions $g_j^*(\tilde{y};\theta,\gamma),\ j = 1,...,J,\ g_j^{*2}(\tilde{y};\theta,\gamma),\ j = 1,...,J$ and for $g_j^*(\tilde{y};\theta,\gamma),\ g_j^*(\tilde{y}_{-h};\theta,\gamma),\ j,k = 1,...,J, h = 1,...,H$. In this expression $\tilde{y}$ denotes the trajectory of the process and $\tilde{y}_{-h}$ denotes this trajectory lagged by h. It is assumed that $sup_T \frac{1}{T} E|\mathcal{B}(Y_{T,t})] < \infty$ [Gourieroux, Jasiak (2022)], where the expectation is taken with respect to the distribution of process $Y_{T.} = (Y_{Tt})$, with varying t.

4. Condition of Near Epoch Dependence [De Jong (1988)]

The functions of the triangular array of random variables $Y_{T,t}, t \leq T, T \geq 1$ are L_2-NED (near epoch dependent), i.e. for $\nu(m) \geq 0$ and $c_{Tt} \geq 0$ and for all $m \geq 0$ and $t \geq 1$

$\sup_{\theta\in\Theta} E[g_j(\tilde{Y}_{T,t},\theta) - E(g_j(\tilde{Y}_{T,t},\theta)|Y_{T,t-m},...,Y_{T,t+m})]^2 \leq c_{T,t}\varphi(m)$

$\sup_{\theta\in\Theta} E[g_j^2(\tilde{Y}_{T,t},\theta) - E(g_j^2(\tilde{Y}_{T,t},\theta)|Y_{T,t-m},...,Y_{T,t+m})]^2 \leq c_{T,t}\varphi(m)$

$\sup_{\theta\in\Theta} E[g_j(\tilde{Y}_{T,t},\theta)g_k(\tilde{Y}_{T,t-h},\theta) - E(g_j(\tilde{Y}_{T,t},\theta)|Y_{T,t-m},...,Y_{T,t+m})E(g_k(\tilde{Y}_{T,t-h},\theta)|Y_{T,t-m},...,Y_{T,t+m})]^2$
$\leq c_{T,t}\varphi(m)$

for all $j,k = 1,...,K,\ h = 1,...,H,\ \varphi(m) \to 0$ as $m \to \infty$ and $\lim\sup_{T\to\infty} \frac{1}{T}\sum_{t=1}^{T} c_{T,t} < \infty$.

From Assumption 4) it follows that the functions $g(Y_{Tt})$ are uniformly integrable mixingales [DeJong (1998)]. Because the NED condition implies a mixingale condition, the weak LLN of Theorem 2, Andrews (1988) can be applied. Then, Theorem 4 of Andrews (1992) implies the uniform weak LLN (U-WLLN).

To summarize, we get the following Proposition:

Proposition B1:

Under the regularity conditions 1 to 4, we have:

$$plim_{T\to\infty}\hat{\Gamma}_T(h;\theta) = \Gamma_0(h;\theta)$$

uniformly in $\theta \in \Theta$ for $h = 1, ..., H$, where $\Gamma_0(h;\theta)$ is evaluated at θ_0, γ_0 and f_0 is the true pdf of the error.

When the functions g are distinguished from their transforms g_a, then conditions 1 and 2 concern g and conditions 3 and 4 concern g_a.

B.2 Central Limit Theorem (CLT) for Triangular Array

We need to introduce additional regularity conditions to justify the expansion (3.13) and the asymptotic normality of the sample autocovariances $\sqrt{T}\hat{\Gamma}(h;\theta_T,\gamma_T,f_0)$ computed under a sequence of local alternatives. To obtain the corresponding CLT, we use the conditional Lindeberg-Feller conditions for martingale difference triangular array [Dvoretski (1970), Brown (1971)] extended to the multivariate case [Kundu et al. (2000), Th. 1.3]. To apply these conditions, we first need to define triangular filtration and the appropriate martingales. We denote by $\mathcal{F}_{T,t}$ the information generated by the array $Y_{T,\tau},\ \tau \leq t$. Then, we consider the different transformations $g_j^*(\tilde{Y}_{T,t};\theta_0,0),\ g_j^*(\tilde{Y}_{T,t};\theta_0,0)g_k^*(\tilde{Y}_{T,t-h};\theta_0,0),\ j,k = 1,...,K,\ h = 1,...,H$. They can be written as a vector $G^*(\tilde{Y}_{T,t};\theta_0,0)$, say. Next, we transform this vector into a multivariate martingale difference array by considering:

$$X_{T,t} = \frac{1}{\sqrt{T}}\{G(\tilde{Y}_{T,t};\theta_0,0) - E_0[G(\tilde{Y}_{T,t};\theta_0,0)|\mathcal{F}_{T,t-1}]\}.$$

The additional regularity conditions are the following:

Regularity Conditions for the CLT

i) The multivariate martingale difference array $X_{T,t}$ has finite second-order moments.

ii) For any vector b of the same dimension as $X_{T,t}$, there exists a matrix Ω such that:

$$\sum_{t=1}^{T} E[(b'X_{T,t})^2|\mathcal{F}_{T,t-1}] \xrightarrow{P} b'\Omega b.$$

iii) Conditional Lindeberg-Feller condition:

$$\sum_{t=1}^{T} E\{(b'X_{T,t})^2 \mathbf{1}_{|b'X_{T,t}|>\epsilon}|\mathcal{F}_{T,t-1}\} \xrightarrow{P} 0, \text{ for any } b \text{ and } \epsilon > 0.$$

These regularity conditions ensure that the sum $S_T = \sum_{t=1}^{T} X_{T,t}$ tends in distribution to the multivariate Gaussian distribution $N(0,\Omega)$. Then we get the asymptotic normality of the estimated autocovariances under the sequence of local alternatives by applying the Slutsky Theorem.

Proposition B2:

Under the sequence of local alternatives and the regularity conditions 1-5, the vectors $vec[\sqrt{T}\hat{\Gamma}_T(h;\theta_T,\gamma_T,f_0)]$ are asymptotically independent, normally distributed with mean $\Delta(h;\theta_0,f_0,\mu,\nu)$ defined in (3.10) and variance-covariance matrix $\Gamma_0(0,\theta_0)\otimes\Gamma_0(0,\theta_0)$

Thus the behavior of the estimated autocovariances differs from its behavior under the null by the presence of the asymptotic bias measured by $\Delta(h;\theta_0,f_0,\mu,\nu)$.

We have introduced a set of regularity conditions to derive the asymptotic behavior of the estimated autocovariances. Let us now explain why this set of conditions is also sufficient to derive the asymptotic behavior of the GCov estimator and of the portmanteau statistic. First, we review the standard expansions under the null hypothesis. Next, we derive their analogues under the sequence of alternatives, before applying the CLT to the estimated autocovariances of a triangular array.

B.3 First-order Expansion of the GCov Estimator under the Null Hypothesis

Below we recall the results under the null hypothesis derived in Gourieroux, Jasiak (2023). Let us consider $H=1$ for ease of exposition. The first-order conditions of the GCov estimator are

$$\frac{\partial Tr\hat{R}^2(1;\theta_j)}{\partial\theta_j}=0,\quad j=1,...,J=dim\theta,$$

Let us define:

$$A(\theta_0)=2\frac{\partial vec\Gamma(1;\theta_0)'}{\partial\theta}[\Gamma(0;\theta_0)^{-1}\otimes\Gamma(0;\theta_0)^{-1}],$$

and

$$J(\theta_0)=-2\frac{\partial vec\Gamma(1;\theta_0)'}{\partial\theta}[\Gamma(0;\theta_0)^{-1}\otimes\Gamma(0;\theta_0)^{-1}]\frac{\partial vec\Gamma(1;\theta_0)}{\partial\theta}.$$

The first-order Taylor series expansion of the GCov estimator is:

$$\sqrt{T}(\hat{\theta_T}-\theta_0)=J(\theta_0)^{-1}A(\theta_0)vec[\sqrt{T}\hat{\Gamma}_T(1;\theta_0)']+o_p(1),\tag{B.3}$$

B.4 Expansion of the Portmanteau Statistic under the Null Hypothesis

The expansion of the test statistic under the null hypothesis is:

$$\hat{\xi}_T(H)=\sum_{h=1}^{H}vec[\sqrt{T}\hat{\Gamma}_T(h,\theta_0,f_0)']'\Pi(h;\theta_0,f_0)vec[\sqrt{T}\hat{\Gamma}_T(h,\theta_0,f_0)']+o_p(1),\tag{B.4}$$

[See Gourieroux, Jasiak, "Generalized Covariance Estimator Supplemental Material" (2023), equation (a.11)], where

$$\begin{aligned}\Pi(h;\theta_0,f_0) &= [\Gamma_0(0,\theta_0,f_0)^{-1}\otimes\Gamma_0(0,\theta_0,f_0)^{-1}] - [\Gamma_0(0,\theta_0,f_0)^{-1}\otimes\Gamma_0(0,\theta_0,f_0)^{-1}]\frac{\partial vec\Gamma(h,\theta_0,f_0)}{\partial\theta'}\\ &\left\{\frac{\partial vec\Gamma(h,\theta_0,f_0)'}{\partial\theta}[\Gamma_0(0,\theta_0,f_0)^{-1}\otimes\Gamma_0(0,\theta_0,f_0)^{-1}]\frac{\partial vec\Gamma(h,\theta_0,f_0)}{\partial\theta}\right\}^{-1}\\ &\times\frac{\partial vec\Gamma(h,\theta_0,f_0)'}{\partial\theta'}[\Gamma_0(0,\theta_0,f_0)^{-1}\otimes\Gamma_0(0,\theta_0,f_0)^{-1}]\end{aligned}$$

Matrix $\Pi(h;\theta_0,f_0)$ satisfies for all $h=1,...,H$ the condition

$$\Pi(h;\theta_0,f_0)V_{asy}[\sqrt{T}\hat{\Gamma}_T(h,\theta_0)']\Pi(h;\theta_0,f_0)=\Pi(h;\theta_0,f_0)$$

where $V_{asy}[\sqrt{T}\hat{\Gamma}_T(h,\theta_0,f_0)']=[\Gamma_0(0,\theta_0,f_0)\otimes\Gamma_0(0,\theta_0,f_0)]$.

This condition means that the matrix $\Pi(h;\theta_0,f_0)$ has an interpretation in terms of an orthogonal projector. Therefore, under the null hypothesis, the quadratic form (A.10) where the $vec(\sqrt{T}\hat{\Gamma}_T(h;\theta_0,f_0))$ are independent identically distributed still follows a chi-square distribution with a reduced degree of freedom.

B.5 Asymptotic Behavior Under the Local Alternatives

Under the regularity conditions 1-6, it is easy to see that expansions similar to (B.3)-(B.4) are still valid under the sequence of local alternatives by using the LLN for triangular arrays and the convergence of order $1/\sqrt{T}$ of the estimated autocovariances that follows from the CLT. For example, we still have the expansion:

$$\hat{\xi}_T(H)=T\sum_{h=1}^{H}\{vec[\sqrt{T}\hat{\Gamma}_T(h;\theta_T,\gamma_T,f_0)]\Pi(h;\theta_0,f_0)vec[\sqrt{T}\hat{\Gamma}_T(h;\theta_T,\gamma_T,f_0)]\}+o_p(1)$$

similar to expansion (B.4) where the vectors $vec[\sqrt{T}\hat{\Gamma}_T(h;\theta_T,\gamma_T,f_0)]$, $h=1,...,H$ are now asymptotically independent with the distribution

$N[\delta(h;\theta_0,f_0,\mu,\nu),\Gamma(0;\theta_0,f_0)\otimes\Gamma(0;\theta_0,f_0)]$.

by the CLT.

Then, under the sequence of local alternatives, the asymptotic distribution of $\hat{\xi}_T(H)$ is a chi-square distribution with the non-centrality parameter λ:

$$\lambda(\theta_0,f_0,\mu,\nu)=\sum_{h=1}^{H}\delta(h,\theta_0,f_0,\mu,\nu)'\Pi(h;\theta_0,f_0)\delta(h,\theta_0,f_0,\mu,\nu)$$

and a degree of freedom equal to the rank of matrix $\Pi(H;\theta_0,f_0)=diag[\Pi(h;\theta_0,f_0)]$, where diag denotes a diagonal matrix.

B.6 The Behavior of the Independence Test under Local Alternatives

The results of Section B.5 can be applied to a null hypothesis $H_0 = (y_t = u_t)$ without parameter θ and other forms of local alternatives.

Let us consider the test of the absence of linear dependence in time series $y_t = u_t, t = 1, ..., T$ against the local alternatives of an autoregressive form. More specifically, we test

$$H_0 : \{\Gamma_0(h) = 0, \forall h = 1, ..., H\} = \{B_1 = \cdots = B_H = 0\},$$

against the local alternatives. The local alternatives can be defined in terms of the autoregressive parameters $B_1, ..., B_H$, or equivalently in terms of the autocovariances $\Gamma(h), h = 1, ..., H$. Thus, the additional parameter γ is not necessarily a scalar. We follow the latter approach with the sequence of local alternatives:

$$H_{1,T} = \{\Gamma_T(h) = \Delta(h)/\sqrt{T},\ h = 1, ..., H\} = \{vec\Gamma_T(h) = \delta(h)/\sqrt{T},\ h = 1, ..., H\},$$

with $\delta(h) = vec\Delta(h)$.

Under the sequence of local alternatives, the estimated autocovariances are asymptotically independent with the asymptotic normal distributions.

$$vec[\sqrt{T}\hat{\Gamma}_T(h)'] \overset{a}{\sim} N[\delta(h), \Gamma(0) \otimes \Gamma(0)].$$

Hence:

$$[\Gamma(0)^{-1/2} \otimes \Gamma(0)^{-1/2}]vec[\sqrt{T}\hat{\Gamma}_T(h)'] \overset{a}{\sim} N[(\Gamma(0)^{-1/2} \otimes \Gamma(0)^{-1/2})\delta(h), Id]$$

It follows that the portmanteau statistic $\hat{\xi}_T(H)$ has asymptotically, under the sequence of local alternatives, a chi-square $\chi^2(K^2H, \lambda)$ distribution with the non-centrality parameter λ, where

$$\lambda = \sum_{h=1}^{H} \delta(h)'[\Gamma(0)^{-1/2} \otimes \Gamma(0)^{-1/2}][\Gamma(0)^{-1/2} \otimes \Gamma(0)^{-1/2}]\delta(h) = \sum_{h=1}^{H} \delta(h)'[\Gamma(0)^{-1} \otimes \Gamma(0)^{-1}]\delta(h), \tag{B.5}$$

is the non-centrality parameter.

Appendix C

Table 6: Empirical power of GCov specification test for MAR(0,1) against the fixed alternative of MAR(1,1) at 5% significance level

ψ	$\gamma = \phi$	T=100			T=200			T=500		
		Uniform	Laplace	t(5)	Uniform	Laplace	t(5)	Uniform	Laplace	t(5)
0.3	0.1	0.0226	0.0528	0.0446	0.0394	0.0672	0.0616	0.0572	0.0758	0.0752
	0.2	0.0276	0.0626	0.0568	0.0720	0.0956	0.0980	0.1986	0.1624	0.1880
	0.3	0.0416	0.0850	0.0866	0.1364	0.1612	0.1748	0.5244	0.3898	0.4162
	0.4	0.0568	0.1080	0.1242	0.1796	0.2286	0.2394	0.6136	0.5280	0.5590
	0.5	0.0604	0.1238	0.1346	0.2174	0.2568	0.2696	0.8000	0.6812	0.7002
	0.6	0.0642	0.1276	0.1332	0.2504	0.3070	0.3216	0.8286	0.7898	0.7938
	0.7	0.0746	0.1460	0.1518	0.2796	0.3614	0.3796	0.8722	0.8644	0.8634
	0.8	0.1016	0.1724	0.1788	0.3404	0.4468	0.4672	0.9092	0.9282	0.9300
	0.9	0.2100	0.2478	0.2552	0.5486	0.6180	0.6174	0.9804	0.9846	0.9862
0.7	0.1	0.0182	0.0500	0.0378	0.0374	0.0702	0.0664	0.1174	0.1224	0.1140
	0.2	0.0296	0.0718	0.0628	0.1012	0.1456	0.1454	0.5346	0.4464	0.4602
	0.3	0.0656	0.1430	0.1526	0.2854	0.3712	0.3764	0.8908	0.8630	0.8654
	0.4	0.1578	0.3120	0.3098	0.5718	0.7040	0.6852	0.9926	0.9950	0.9908
	0.5	0.3392	0.5700	0.5490	0.8318	0.9316	0.9196	1	1	0.9996
	0.6	0.6034	0.8144	0.7848	0.9690	0.9914	0.9898	1	1	1
	0.7	0.8528	0.9424	0.9370	0.9990	0.9996	1	1	1	1
	0.8	0.9680	0.9882	0.9896	1	1	1	1	1	1
	0.9	0.9970	0.9980	0.9990	1	1	1	1	1	1

The following Tables 7 and 8 provide additional simulation results on the NLSD and GCov specification tests, respectively. Table 7 provides the results on the empirical size of the NLSD test for local alternatives and δ increasing from 0 to 0.9. Table 8 shows the empirical size of the GCov specification test for fixed alternatives and different values of ψ.

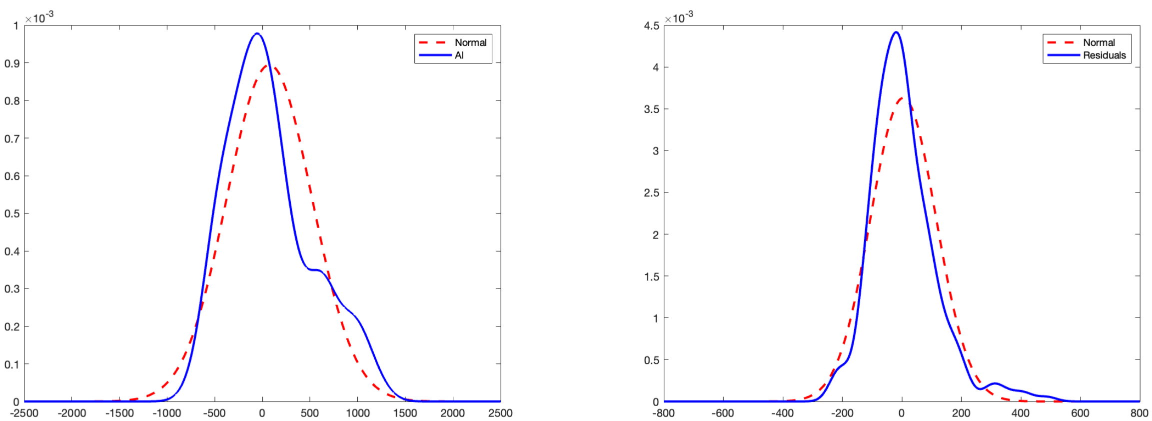
Table 7: Test of the absence of (non)linear dependence (MAR(0,0)) against local MAR(0,1) alternatives at 5% significance level: size and size-adjusted power

$\gamma_T = \frac{\delta}{\sqrt{T}}$	T=100			T=200			T=500		
	Uniform	Laplace	t(5)	Uniform	Laplace	t(5)	Uniform	Laplace	t(5)
$\frac{0}{\sqrt{T}}$	0.0414	0.0484	0.0512	0.0450	0.0502	0.0518	0.0496	0.0540	0.0480
$\frac{0.1}{\sqrt{T}}$	0.0498	0.0494	0.0470	0.0496	0.0490	0.0504	0.0512	0.0494	0.0508
$\frac{0.2}{\sqrt{T}}$	0.0498	0.0488	0.0460	0.0502	0.0496	0.0496	0.0528	0.0492	0.0514
$\frac{0.3}{\sqrt{T}}$	0.0516	0.0474	0.0464	0.0506	0.0524	0.0486	0.0544	0.0504	0.0540
$\frac{0.4}{\sqrt{T}}$	0.0526	0.0486	0.0474	0.0530	0.0536	0.0500	0.0564	0.0520	0.0564
$\frac{0.5}{\sqrt{T}}$	0.0574	0.0502	0.0486	0.0556	0.0568	0.0524	0.0606	0.0550	0.0602
$\frac{0.6}{\sqrt{T}}$	0.0596	0.0530	0.0492	0.0606	0.0610	0.0548	0.0648	0.0584	0.0642
$\frac{0.7}{\sqrt{T}}$	0.0650	0.0552	0.0520	0.0664	0.0662	0.0606	0.0710	0.0620	0.0710
$\frac{0.8}{\sqrt{T}}$	0.0714	0.0580	0.0564	0.0756	0.0742	0.0652	0.0772	0.0664	0.0774
$\frac{0.9}{\sqrt{T}}$	0.0804	0.0640	0.0622	0.0838	0.0812	0.0714	0.0860	0.0758	0.0848

The first row ($\psi = 0$) shows the empirical size of test and the remaining rows show the power with respect to local alternatives with $\gamma_T = \psi_T = \frac{\delta}{\sqrt{T}}$.

Table 8: Empirical size of GCov specification test for the null hypothesis of MAR(0,1) at 5% significance level

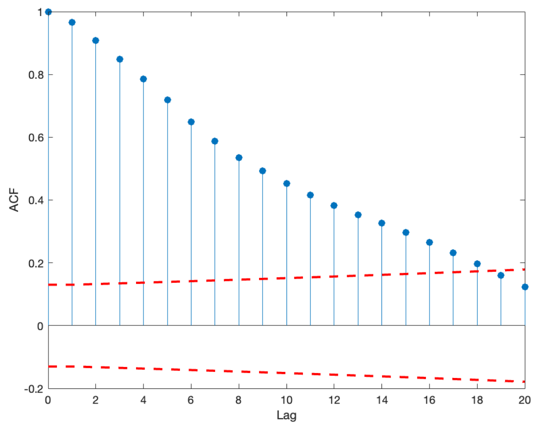
ψ	T=100			T=200			T=500		
	Uniform	Laplace	t(5)	Uniform	Laplace	t(5)	Uniform	Laplace	t(5)
0.1	0.0212	0.0404	0.0378	0.0342	0.0538	0.0514	0.0408	0.0538	0.0544
0.2	0.0216	0.0414	0.0390	0.0344	0.0548	0.0534	0.0414	0.0558	0.0560
0.3	0.0224	0.0454	0.0386	0.0348	0.0566	0.0534	0.0406	0.0544	0.0560
0.4	0.0214	0.0460	0.0390	0.0344	0.0560	0.0522	0.0414	0.0548	0.0540
0.5	0.0194	0.0450	0.0382	0.0326	0.0550	0.0518	0.0414	0.0548	0.0540
0.6	0.0196	0.0428	0.0348	0.0318	0.0536	0.0492	0.0426	0.0560	0.0544
0.7	0.0200	0.0400	0.0338	0.0298	0.0528	0.0468	0.0408	0.0552	0.0528
0.8	0.0202	0.0392	0.0316	0.0296	0.0512	0.0454	0.0396	0.0532	0.0500
0.9	0.0178	0.0384	0.0310	0.0278	0.0508	0.0428	0.0350	0.0494	0.0498



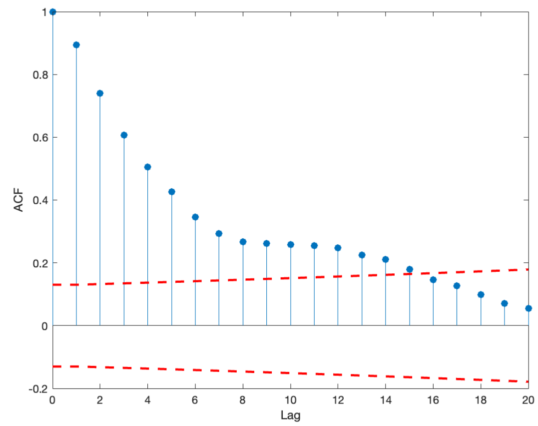
(a) Demeaned-Detrended Aluminum price

(b) MAR(1,1) residuals

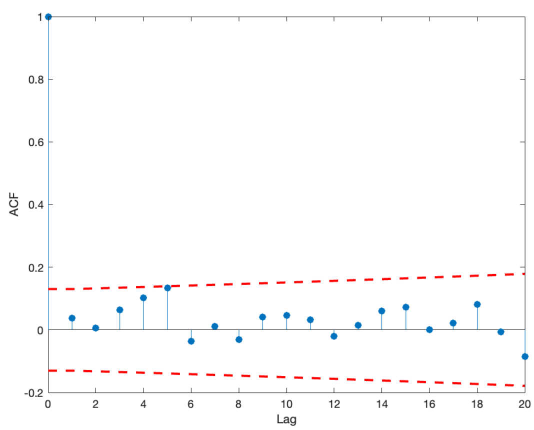
Figure 6: Densities of demeaned Aluminum price and MAR(1,1) residuals, compared with the Normal density



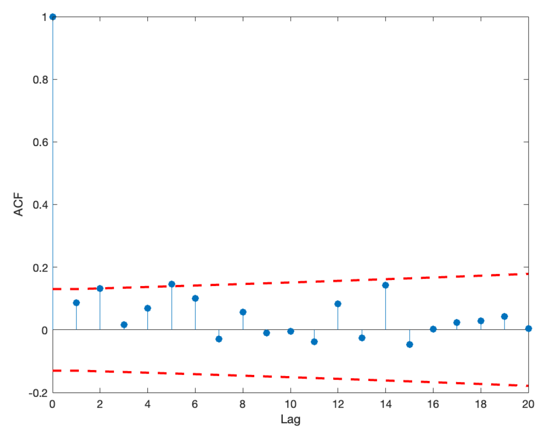
(a) ACF of the series



(b) ACF of the series square



(c) ACF of the residuals



(d) ACF of the square of the residuals

Figure 7: ACF of Aluminum prices and squared prices (panels a and b) and MAR(1,1) residuals and squared residuals (panels c and d)

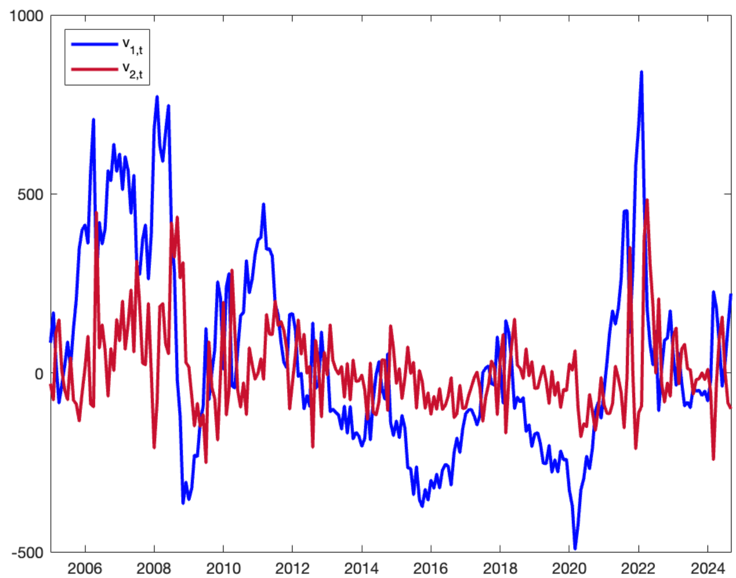


Figure 8: MAR(1,1) causal-noncausal components