Optimal Positional Momentum and Liquidity Management

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Abstract

This paper introduces new positional investment strategies that maximize investors' positional utility from holding assets with high expected future return and liquidity ranks. The optimal allocation vectors provide new investment strategies, such as the optimal positional momentum portfolio, the optimal liquid portfolio and the optimal mixed portfolio that combines high return and liquidity ranks. The future ranks are predicted from a bivariate panel VAR model with time varying autoregressive parameters. We show that there exists a simple linear relationship between the time varying autoregressive parameters of the VAR model and the auto-and cross-correlations at lag one of the return and volume change series of the SPDR. Therefore the autoregressive VAR parameters can be easily updated at each time, which simplifies the implementation of the proposed strategies. The new optimal allocation portfolios are shown to perform well in practice, both in terms of returns and liquidity.

Keywords:Optimal Positional Momentum, Optimal Positional Liquidity, CARA Utility Function, Bivariate Ranks, Panel VAR, SPDR

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1 Introduction

Using the utility function as an agent objective function is the foundation for the portfolio selection under uncertainty. According to the literature, utility function measures the investor's relative preference for different levels of wealth. One of the advantages of the utility-based strategy is that it eliminates the arbitrary cut-off point of top 5%, or top 10% of assets to be included in a portfolio. In the portfolio management literature, the investor maximizes his/her expected utility function based on wealth or portfolio return [see Brennan and Torous (1999), Das and Uppal (2004) and Gourieroux and Monfort (2005)][1], while in this paper the investor is assumed to maximize a CARA (Constant Absolute Risk Aversion) utility function of future position of the assets (ranks of assets). In this respect, we follow the approach of Gagliardini, Gourieroux, Rubin (2019) who introduce a positional utility, which is an increasing function of future asset return positions rather than of future portfolio returns.

This paper introduces new positional investment strategies that maximize investors' utility from holding assets with high expected future ranks in return and liquidity. This approach allows us to determine the optimal allocations that select assets with respect to their expected future returns and liquidity ranks, where the latter ones are measured by changes in traded volumes. An optimal allocation vector is also derived for a mixed portfolio of assets with the highest combined ranks of returns and liquidity. The new allocation strategies are called the optimal positional momentum portfolio, the optimal positional liquid portfolio and the optimal positional mixed portfolio, respectively. The new optimal allocations that maximize the positional utility function arise as extensions of a naive equally weighted portfolio that account for serial dependence in the returns and volume change ranks as well as for their co-movements. We show that returns on the new optimal portfolios are comparable both theoretically and empirically with the naive equally weighted portfolio as well as with the traditional momentum strategies with look-back and holding periods of various length.

The future ranks of returns and volume changes are predicted from a bivariate panel Vector Autoregressive (VAR) model. In order to adapt the ranks to the VAR dynamics, the bivariate series of return and volume change ranks are first transformed into Gaussian

[1] Von Neumann and Morgenstern (1994) show that, a rational investor selects the optimal feasible investment by maximising the expected utility of wealth.

ranks. We observe that the autoregressive parameters of the VAR model display variation over time. To accommodate that variation, we consider a time varying parameter VAR model and propose two methods that allow an investor to update the VAR parameters at each investment time. The first method consists in re-estimating the model at each time by rolling over a fixed window of observations. The second method exploits the relationship between the autoregressive coefficients of the VAR model and the series of auto-and cross-correlations at lag 1 of returns and volume changes of the SPDR (Standard Poor's Depositary Receipts). The SPDR is an Exchange Traded Fund (ETF), i.e. a regularly updated portfolio mimicking the evolution of the $S\&P$ 500 returns[2]. More specifically, we show that the future values of autoregressive VAR coefficients can be predicted from simple linear functions of the current auto- and cross-correlations at lag 1 of SPDR's return and volume changes. These linear functions are easy to compute and simplify the investment procedure as they eliminate the need for re-estimating the panel VAR model by rolling. In the proposed approach, the time varying parameters are considered predetermined. We show heuristically that the approach can be extended to a random coefficient framework, where the autoregressive VAR coefficients are considered as fixed functions of random factors, which are the auto and cross-correlation estimators with their known asymptotic distributions.

In the financial literature the risk-return trade-off or the risk–reward shows the amount of return gained on an investment correspond to the amount of undertaken risk. Modern Portfolio Theory (MPT) assume that investors are risk averse and many literature show that the more return sought, the more risk that must be undertaken [see Breen, Glosten, and Jagannathan (1989), Nelson (1991), Glosten, Jakannatha and Runkle (1993), Brandt and Kang (2004), etc.]. It means that, given two portfolios with the same expected return, investors will prefer the less risky one and an investor will take more risk only for higher expected returns. On the other hand, this trade-off is not the same for all investors, different investors will evaluate the trade-off differently based on individual risk

[2]Beaulieu and Morgan (2000) studied the high-frequency relationships between the S&P 500 Index and the SPDR by using minute-by-minute data for November 1997 through February 1998. They showed that the SPDR did not track the index perfectly. Peng Xu (2014) checked the mimicking performance of the SPDR in two ways: first he examined the relation between relative price change of the SPDR and the relative change of the index and second studied the relation between holding period return of the SPDR and the return on the index. He showed that in a linear static analysis the SPDR mimics the index pretty well, since the historical correlation coefficient between the two return series is 0.98. He also showed that both series will have similar dynamic features, as long as linear dynamics are considered

aversion characteristics. Computing the level of an individual's risk aversion is the most difficult question since the answer is subjective.[3] In many literature the risk aversion is considered constant since it allows models to reach precise and relatively simple formulas for relationships between variables[4]. In this paper, we consider the CARA utility function with a constant risk aversion while the investor can adjust the portfolio to the current market conditions by changing the risk aversion coefficient to invest more or less aggressively.

The paper is organized as follows. Section 2 introduces the panel VAR model and its parameter estimates based on the entire sample. It also provides the evidence of time variation of the autoregressive coefficients and extends the model to a time varying parameter VAR model. Section 3 documents empirically and establishes the linear relationship between the auto- and cross-correlations of the return and volume change series of SPDR and the series of autoregressive coefficients of the VAR model. Section 4 derives the optimal allocation vectors from maximizing the positional CARA utility functions of expected ranks of return and volume changes that lead to the optimal momentum, liquid and mixed portfolios. Section 5 presents the empirical results. Section 6 concludes the paper. Additional results are gathered in Appendices A, B, C and D.

2 The Cross-Sectional Gaussian Ranks Model

2.1 The Ranks

This chapter examines the dynamics of Gaussian ranks of return and trade volume changes computed from 1330 stocks observed monthly over the period of April 1999 to October 2016. The ranks are defined in Chapter two, Section 3 as follows:

$$u_{i,t} = \Phi^{-1}(\hat{F}_t^r(r_{it})) \qquad t = 1, \cdots, T; \quad i = 1, \cdots, n, \tag{2.1}$$

[3]There are some tests help determine what is the most appropriate risk for investors. The PASS test by W.G. Droms (1988), the Baillard, Biehl Kaiser (1986) test, classifies investors in order from "confident" to "anxious" and "careful" to "impetuous", while Barnewal (1987) considered just two types of investors passive and active investors.

[4]Chou (1988) showed that the risk attitude parameter stay stable for correlative periods of time, Safra and Segal (1998) defined the invariant preference relation between outcomes of two distributions as the constant risk aversion and Quiggin and Chambers (2004) show the constancy of the risk aversion since the investor attitude is strongly linked with the family of generalized expected utility preferences.

$$v_{i,t} = \Phi^{-1}(\hat{F}_t^{tv}(tv_{it})) \qquad t = 1, \cdots, T; \quad i = 1, \cdots, n, \tag{2.2}$$

where $u_{i,t}$ is the Gaussian rank of return $(r_{i,t})$, $v_{i,t}$ is the Gaussian rank of trade volume change $(tv_{i,t})$, Φ is the cumulative distribution function (c.d.f) of the standard Normal, Φ^{-1} is its inverse, i.e. the quantile function of the standard Normal and $\hat{F}_t^r, \hat{F}_t^{tv}$ are the cross-sectional empirical cumulative distribution functions of return and trade volume changes at date t, respectively.

2.2 The Model

The positional portfolio strategy is about finding the optimal allocation based on the future position of all equities in the portfolio. To predict the future positions, we define a joint dynamic model of ranks of return and trade volume changes $(u_{it}, v_{it} : i = 1, \cdots, n, t = 1, \cdots, T)$. The joint dynamics of the two rank series can be represented by a Vector Autoregressive model of order one (VAR(1)) as follow:

$$\begin{pmatrix} u_{it} \\ v_{it} \end{pmatrix} = \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix} \begin{pmatrix} u_{i,t-1} \\ v_{i,t-1} \end{pmatrix} + \Sigma^{1/2} \begin{pmatrix} e_{1,it} \\ e_{2,it} \end{pmatrix} \qquad t = 2, \cdots, T;\ i = 1, \cdots, n, \tag{2.3}$$

where $R = \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix}$ is the matrix of autoregressive coefficients , Σ represents the conditional variance matrix and the idiosyncratic disturbance terms $(e_{1it}, e_{2,it})$ are serially independent and identically (i.i.d.) standard Normal distributed. The autoregressive matrix R is assumed to have eigenvalues with modulus less than one to ensure the stationarity of the process. The ranks are marginally standard Normally distributed with the marginal variance of the ranks $\begin{pmatrix} 1 & \eta \\ \eta & 1 \end{pmatrix}$. Let us introduce an additional assumption as follow:

Assumption 1, The marginal variance of ranks is an identity matrix.

The above assumption implies that $\eta = 0$ [5]. Moreover, it constraints the error variance matrix Σ as follows:

[5] This assumption is not very stringent. In Chapter 2, we have empirically documented that $\hat{\eta}$ is small and tends to 0 at the end of the sampling period.

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = R \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} R' + \Sigma. \tag{2.4}$$

From equation (2.4) we can compute the matrix Σ as follows:

$$\Sigma = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - R \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} R' = Id - RR', \tag{2.5}$$

where Id is a 2×2 identity matrix. Matrix Σ depends on the autoregressive coefficients of the VAR(1) model:

$$\Sigma = \begin{pmatrix} 1 - \rho_{11}^2 - \rho_{12}^2 & -\rho_{11}\rho_{21} - \rho_{12}\rho_{22} \\ -\rho_{11}\rho_{21} - \rho_{12}\rho_{22} & 1 - \rho_{21}^2 - \rho_{22}^2 \end{pmatrix} = \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2. \end{pmatrix} \tag{2.6}$$

The VAR(1) model (2.3) can be rewritten as follows:

$$\begin{pmatrix} u_{it} \\ v_{it} \end{pmatrix} = \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix} \begin{pmatrix} u_{i,t-1} \\ v_{i,t-1} \end{pmatrix} + \begin{pmatrix} \epsilon_{1,it} \\ \epsilon_{2,it} \end{pmatrix}, \qquad t = 2, \cdots, T;\ i = 1, \cdots, n, \tag{2.7}$$

where error vectors $(\epsilon_{1,it}, \epsilon_{2,it})$ are jointly normally distributed with mean 0 and variance Σ. The marginal densities of the error terms are:

$$\begin{aligned} \epsilon_{1,it} &\sim N(0, \sigma_1^2), \\ \epsilon_{2,it} &\sim N(0, \sigma_2^2), \end{aligned} \tag{2.8}$$

where $\sigma_1^2 = 1 - \rho_{11}^2 - \rho_{12}^2$, $\sigma_2^2 = 1 - \rho_{21}^2 - \rho_{22}^2$ and $cov(\epsilon_{1,it}, \epsilon_{2,it}) = \sigma_{12} = -\rho_{11}\rho_{21} - \rho_{12}\rho_{22}$. The parameters of model (2.7) are estimated by the maximum likelihood method with the following objective function that is maximized with respect to the autoregressive parameters $(\rho_{11}, \rho_{12}, \rho_{21}$ and $\rho_{22},)$:

$$
\begin{aligned}
logL = \sum_{i=1}^{N}\sum_{t=2}^{T}\Big\{ - log(2\pi) - \frac{1}{2}log(|Id - RR'|) - \frac{1}{2}\Big[\begin{pmatrix} u_{it} \\ v_{it}\end{pmatrix} - R\begin{pmatrix} u_{it-1} \\ v_{it-1}\end{pmatrix}\Big]' \\
\cdot (Id - RR')^{-1}\Big[\begin{pmatrix} u_{it} \\ v_{it}\end{pmatrix} - R\begin{pmatrix} u_{it-1} \\ v_{it-1}\end{pmatrix}\Big]\Big\}
\end{aligned} \tag{2.9}
$$

Table 1 shows the results of the maximum likelihood estimation from ranks of all 1330 stocks over the entire sampling period 1999-2016.

Table 1: Estimated VAR(1) Model for 1330 Stocks

Coefficients	**Values**	**S-D**	**Confidence Interval**
ρ_{11}	-0.024***	0.002	(-0.029 , -0.020)
ρ_{12}	0.012***	0.002	(0.008 , 0.016)
ρ_{21}	-0.010***	0.002	(-0.014 , -0.004)
ρ_{22}	-0.354***	0.001	(-0.357 , -0.351)

$^{***}p < 0.01$, $^{**}p < 0.05$, $^{*}p < 0.1$

The empirical results show that all coefficients of the model are statistically significant. The estimated signs of the autoregressive coefficients suggest that:

1) low ranks of past returns and high ranks of past volume changes tend to increase the current ranks of returns,

2) low ranks of past returns and low ranks of past volume changes tend to increase the current ranks of volume changes.

An important characteristic of a VAR process is its stationarity. A stationary VAR model has time-invariant mean, variance, and covariance structure. In practice, the stationarity of an empirical VAR process can be analyzed by calculating the eigenvalues of the autoregressive coefficient matrix ($\hat{R}$). The computed eigenvalues of ($\hat{R}$) are -0.358 and -0.025. Since both eigenvalues are of modulus less than one, we can conclude that the VAR(1) model is stationary.

Given that the sampling period is long, one can be concerned about the stability of the estimated parameters. Therefore, we re-estimate the equation (2.7) by rolling with the window of 108 months ($\simeq$ 9 years). The rolling estimation yields the estimates of the following VAR(1) with time varying coefficients:

$$\begin{pmatrix} u_{it} \\ v_{it} \end{pmatrix} = \begin{pmatrix} \rho_{11,t} & \rho_{12,t} \\ \rho_{21,t} & \rho_{22,t} \end{pmatrix} \begin{pmatrix} u_{i,t-1} \\ v_{i,t-1} \end{pmatrix} + \begin{pmatrix} \epsilon_{1,it} \\ \epsilon_{2,it} \end{pmatrix}, \qquad t = 2, \cdots, T;\ i = 1, \cdots, n, \tag{2.10}$$

where the error variances are time varying as well: $\sigma_{1t}^2 = 1 - \rho_{11,t}^2 - \rho_{12,t}^2$, $\sigma_{2t}^2 = 1 - \rho_{21,t}^2 - \rho_{22t}^2$ and $cov(\epsilon_{1,it}, \epsilon_{2,it}) = \sigma_{12t} = -\rho_{11,t}\rho_{21,t} - \rho_{12,t}\rho_{22,t}$.

Figures 1 and 2 show the time series of autoregressive coefficients of model (2.10) estimated by rolling over the period: March 2008 - September 2016. We observe that there is some variation in $\hat{\rho}_{11,t}$, which is more pronounced than in $\hat{\rho}_{12,t}$. Coefficient $\hat{\rho}_{11,t}$ varies between -0.015 and -0.005 and coefficient $\hat{\rho}_{12,t}$ varies between 0 and 0.01. Coefficient $\hat{\rho}_{21,t}$ takes lower values and fluctuates between -0.015 and -0.025. Coefficient $\hat{\rho}_{22,t}$ varies around -0.18.

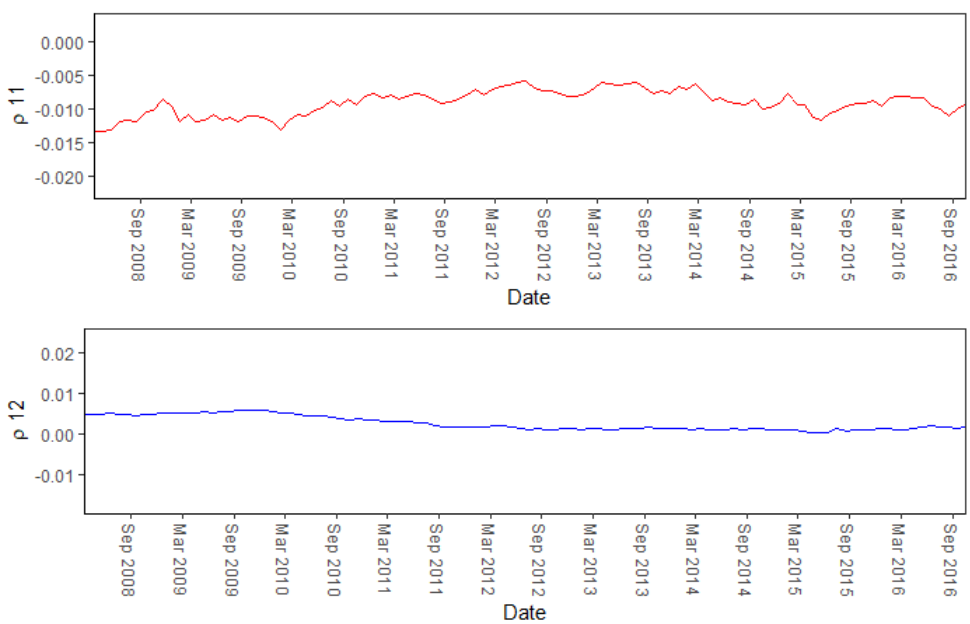

Figure 1: Time Series of $\hat{\rho}_{11,t}, \hat{\rho}_{12,t}$

Figure 1 shows the time series of coefficients $\hat{\rho}_{11,t}, \hat{\rho}_{12,t}$, which are obtained by re-estimating model (2.10) by rolling with the window of 108 months ($\simeq$ 9 years).

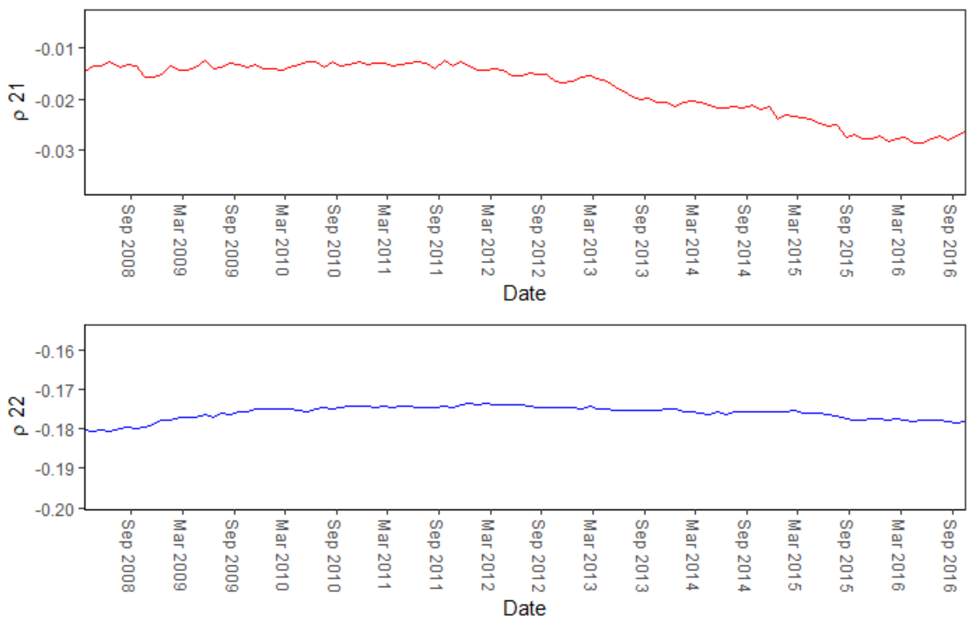

Figure 2: Time Series of $\hat{\rho}_{21,t}, \hat{\rho}_{22,t}$

Figure 2 shows the time series of coefficients $\hat{\rho}_{21,t}, \hat{\rho}_{22,t}$, which are obtained by re-estimating the model (equation (2.10)) by rolling with the window of 108 months ($\simeq$ 9 years).

Let $\hat{R}_t,\ t = 1, ..., T$ denote the time series of matrices of time varying autoregressive coefficients from model (2.10). The eigenvalues of matrices $\hat{R}_t,\ t = 1, ..., T$ computed over $t = 1, ..., T$ are of modulus less than one, indicating that the time varying coefficient VAR(1) model is stationary. We also compute the eigenvalues of the constrained matrices $\hat{\Sigma}_t = Id - R_t R_t',\ t = 1, ..., T$ that are positive at all times $t = 1, ..., T$.

In practice, the rolling estimation of a panel VAR(1) model can be difficult. Therefore, in next Section we explore an alternative approach, where the autoregressive coefficients can be modelled as simple linear functions of time varying factors that are easy to compute.

3 Dynamic Autoregressive Coefficient Model

The stock prices behavior is reflected by the dynamics of stock market indexes such as the $S\&P500$ and by the prices of its mimicking portfolio, called the $SPDR$ (Standard & Poor's Depository Receipts) quoted on NYSE with ticker SPY.

3.1 Standard & Poor's Depository Receipts (SPDR) as Market Factor

The Standard and Poor's Depository Receipt (SPDR),[6] is an exchange traded fund which holds all of the S&P 500 Index stocks and is designed to reflect the price and yield performance of the S&P 500 Index. The SPDR, first issued by the State Street Global Advisors' investment management group (SSGA) and is traded on the American Stock Exchange (AMEX) since 1993. The SPDR index fund is designed to track the S&P 500 stock market index.

The aim behind this ETF is to provide an investment vehicle that at least roughly produces returns in line with the S&P 500 Index. Unlike mutual funds, the SPDR's trust shares are not created for investors at the time of their investment. In fact, they have a fixed number of shares that are bought and sold on the open market to align their holdings with the S&P 500 index. The S&P 500 index itself is composed of U.S. big companies across all Global Industry Classification Standard (GICS) sectors with a market capitalization of $5 billion or greater. Some literature showed that the SPDR is not mimicking S&P 500 perfectly [see Beaulieu and Morgan (2000)]. while some studies show that the SPDR is mimicking S&P 500 in a linear analysis. For instance, Peng Xu (2014) showed that in a linear dynamics analysis the SPDR and S&P 500 has similar dynamic features while. Since the SPDR is designed to reflect the price and yield performance of the SP 500 Index, it can be considered as the pulse of the U.S. equity market or a common factor that encompasses the effects of all news and events on the stock market.

The SPDR is consistently one of the high volume trading vehicles in the U.S. exchanges[7]. Many investors and hedge funds use this fund because it represents the S&P 500 index and by a single purchase, they will have exposure to a wide range of large U.S. companies. Not only the volume but also its good price movement make the SPDR attractive to traders.

Figure 3 shows the relationship between the monthly returns on $SPDR$ and $S\&P500$ recorded over the period April 1999 to October 2016[8]. We observe that these returns are moving in parallel and are both fluctuating roughly between -0.1 to 0.1. There are periods when the volatility of SPDR's returns is higher than the volatility of the return on S&P

[6] Often referred to as the "spider", and its symbol in the market is SPY

[7] Peng Xu (2014) showed that, the average daily trading volume from Jan, 2001 to Dec, 2005 is over 38 million shares and the average trading value per day is over 4 billion

[8] The returns of SPDR and S&P 500 are computed as log return and the dividends haven't been considered in the return.

500. For instance, on February, 2000, September, 2001, August, 2002, October, 2008 or September 2011 the returns of SPDR declined more than the returns on S&P 500. Also at the beginning of years 2009 and 2012, July 2013 and at the end of 2014 the returns of SPDR increased more than the returns on S&P 500.

The historical correlation between the returns on SPDR and on S&P500 is 0.66 and the historical correlation between the squared returns of SPY and S&P 500 is 1.34.[9], suggesting that the SPDR mimics the index rather well as far as a linear static analysis is concerned. Applying a simple linear regression[10], also showed that, these two historical correlations are both statistically significant. From what We observe in Figure 3 and also from the linear regressions' results, we can conclude that the returns on SPDR approximate the S&P 500 returns very closely. Therefore, the returns on the SPDR can be considered as a proxy for the market portfolio return.

Figure 3: SPDR and S&P500 Returns

Figure 3 shows the time series of S&P 500 and SPDR's returns from April 1999 to October 2016.

[9]Which is corresponds to Peng Xu (2014) who showed the positive historical correlation coefficient between the two return series.

[10]A simple linear regression model between SPDR and S&P 500 returns has been estimated as $r_{SPDR} = a_0 + a_1 r_{S\&P500} + e$, and between the squared returns as $r^2_{SPDR} = a_0 + a_1 r^2_{S\&P500} + e$. Where r_{SPDR}, r_{SP} are the return of SPDR and S&P500 respectively, r^2_{SPDR}, r^2_{SP} are the squared return of SPDR and S&P 500, a_0, a_1 are the constant and the coefficient respectively and e is the error term.

3.2 Relation Between The Returns and Trade Volume Changes of SPDR

Let us now consider the series of $SPDR$ returns and trade volume changes recorded monthly between April 1999 and October 2016. The trade volume is defined as the total quantity of shares traded per month. The log return and the log volume changes are calculated as follows:

$$\begin{aligned} r_t^S &= ln(\frac{P_t^S}{P_{t-1}^S}), \qquad & t = 1, \cdots, T \\ tv_t^S &= ln(\frac{TV_t^S}{TV_{t-1}^S}), \qquad & t = 1, \cdots, T, \end{aligned} \tag{3.11}$$

where P_t^S, P_{t-1}^S are the prices of SPDR at times t and $t-1$, TV_t^S, TV_{t-1}^S are the trade volume changes of SPDR at times t and $t-1$.

A simple way to determine whether there exists a relationship between the series of SPDR returns and trade volume changes, is to examine the cross-correlation function. Figure 4 shows the cross-correlation function of returns and trade volume changes of SPDR. We observe that the cross-correlation at lag one is significant. Hence, past trade volume changes can help predict the current returns. We also detect a significant negative contemporaneous correlation between the returns and trade volume changes of SPDR.

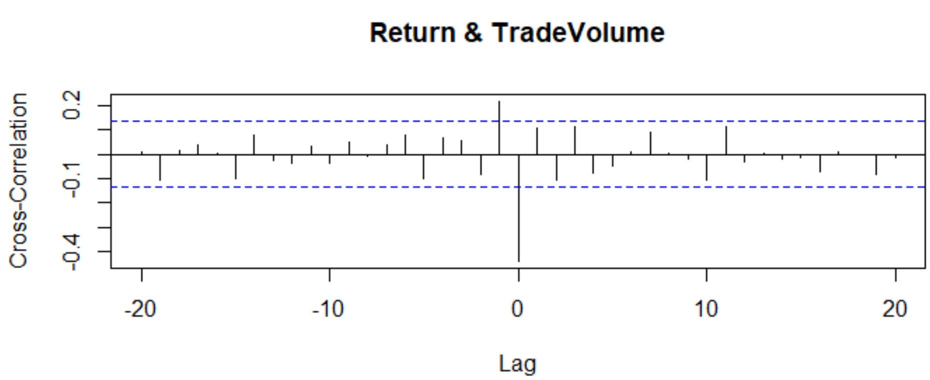

Figure 4: SPDR: Cross-Correlation Function of r_t^S and tv_t^S

Figure 4 shows the cross-correlation function of returns and trade volume changes of SPDR. There is significant correlation at lags 0 and one.

Figure 5 illustrates the contemporaneous correlation in a regression of SPDR trade volume changes on the returns i.e. r_t^S on tv_t^S. The regression line has a negative slope which is consistent with the negative contemporaneous correlation in Figure 4. Hence, a high positive return on SPDR is associated with a high negative trade volume change at time t

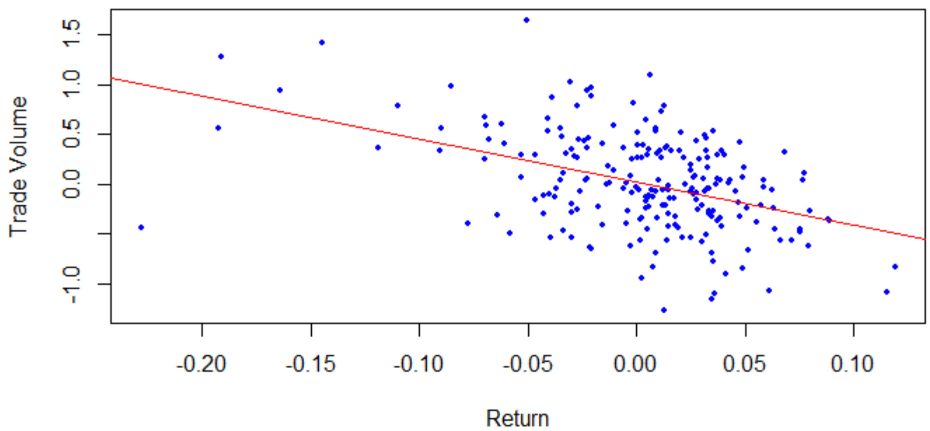

Figure 5: Regression line of r_t on tv_t of SPDR

Figure 5 shows the regression line for returns and trade volume changes of SPDR at time t. The regression line has a negative slope, which shows a negative contemporaneous correlation between return and trade volume changes of SPDR.

Table 2, shows the result of the linear regression of SPDR's trade volume over its return. The correlation between trade volume and return is strongly statistically significant and shows the negative relation between these two variables. It means that if the return increase the contemporaneous trade volume would decrease.

3.3 Comparing Return and Liquidity Persistence: SPDR and Stock Ranks

We have shown that the dynamics of returns on SPDR mimic the dynamics of market returns and the SPDR returns are correlated with the SPDR's trade volume changes.

Table 2: Linear Regression of Correlation Between SPDR's Trade Volume and Return

Coefficients	**Values**	**S-D**	**Confidence Interval**
Intercept	0.023	0.030	(-0.037 , 0.083)
Correlation	-4.335***	0.613	(-5.562 ,-3.108)

$^{***}p < 0.01$, $^{**}p < 0.05$, $^{*}p < 0.1$

Note: Table 2, shows the results of linear regression of SPDR's trade volume changes over its return.

Moreover, the liquidity of SPDR is the liquidity of an asset with a return equal to the market return.

Let us now explore whether the persistence and cross-correlation of returns and trade volume changes of SPDR is similar to rank persistence in all stocks in our sample. That persistence on average over the entire sampling period is approximated by the estimated autoregressive coefficients of stock return and liquidity ranks $\hat{\rho}_{ij}, i, j = 1, 2$ of model (2.10) reported in Table 1. The time varying stock persistence at each time t is approximated by the series of time-varying autoregressive coefficients $\hat{\rho}_{11,t}, \hat{\rho}_{12,t}, \hat{\rho}_{21,t}, \hat{\rho}_{22,t}$ estimated by rolling and displayed in Figures 1 and 2. We proceed with a dynamic analysis and compare these four series with the series of sample auto- and cross-correlations at lag one of r_t^S and tv_t^S, both estimated by rolling with a window of 108 months ($\sim$ 9 years).

Let the dynamic sample autocorrelations at lag 1 be denoted by $AC(r^S)_t$ and $AC(tv^S)_t$ for returns and trade volume changes, respectively. The dynamic sample cross-correlations at lag 1 between r_t^S and tv_{t-1}^S are denoted by $CC(r^S, tv^S)_t$. The sample cross-correlations between $tv_t^S$and r_{t-1}^S are denoted by $CC(tv^S, r^S)_t$. The distributional properties of these time series are examined and compared in Appendix A, which displays their histograms and non-parametric normal density estimates.

Table 3 below shows the means, modes and standard deviations (S.D.) of the time series of $AC(r^S)_t$, $AC(tv^S)_t$, $CC(r^S, tv^S)_t$ and $CC(tv^S, r^S)_t$ in comparison with the autoregressive coefficient series ($\hat{\rho}_{jk,t},\ i, j = 1, 2,\ t = 1, ..., T$). The mean of the sample auto-correlations of SPDR returns and the mean and mode of $\hat{\rho}_{11t}$ are negative. The mean and mode of cross-correlations of r_t^S, tv_{t-1}^S and $\hat{\rho}_{12t}$ have the same sign and are positive while they are bigger for the cross-correlations of r_t^S, tv_{t-1}^S. The mean and mode of cross-correlations of tv_t^S, r_{t-1^S} are positive while the are negative for $\hat{\rho}_{21t}$. Both sample auto-correlation at lag one of SPDR's trade volume changes and $\hat{\rho}_{22t}$ have negative mean and mode.

Table 4 shows the results of the t-test of equality of means of these time series. The t-test of the equality of means of sample auto- and cross-correlation of SPDR and the autoregressive coefficients of the VAR(1) model reject the null hypothesis except for the auto-correlation at lag one of r_t^S and $\hat{\rho}_{11t}$.

Table 3: Summary Statistics for Cross- and Auto- Correlation of SPDR and Autoregressive Coefficients $\hat{\rho}_{jk,t}$

Coefficients	**Mean**	**Mode**	**S.D.**
$AC(r^S)_t$	-0.0001	0.0501	0.0846
$\hat{\rho}_{11,t}$	-0.0090	-0.0084	0.0000
$CC(r^S, tv^S)_t$	0.0647	0.0037	0.1020
$\hat{\rho}_{12,t}$	0.0026	0.0013	0.0016
$CC(tv^S, r^S)_t$	0.2494	0.2991	0.0691
$\hat{\rho}_{21,t}$	-0.0179	-0.0138	0.005
$AC(tv^S)_t$	-0.5089	-0.5176	0.0182
$\hat{\rho}_{22,t}$	-0.1758	-0.1748	0.0016

Note: Table 3, shows Summary Statistics for Cross-Correlation ($CC(r^S, tv^S)_t$,$CC(tv^S, r^S)_t$) and Auto-Correlation ($AC(r^S)_t$,$AC(tv^S)_t$) of SPDR and the Autoregressive Coefficients $\hat{\rho}_{jk,t}$

Table 4: T-Test of Equality of the Means

Null Hypothesis	**P-Value**
Mean(AC(r^S)$_t$)=Mean($\hat{\rho}_{11t}$)	0.155
Mean(CC(r^S, tv^S)$_t$)=Mean($\hat{\rho}_{12t}$)	0.000
Mean(CC(tv^S, r^S)$_t$)=Mean($\hat{\rho}_{21t}$)	0.000
Mean(AC(tv^S)$_t$)=Mean($\hat{\rho}_{22t}$)	0.000

Note: Table 4, shows the t-test results of equality of the mean of Auto- and Cross- Correlation of SPDR and the Autoregressive Coefficients $\hat{\rho}_{jk,t}$

Figures 6-9 below illustrate and compare the dynamics of the series of sample auto-and cross-correlations of SPDR with the autoregressive coefficient dynamics. The right panels show the estimated time varying autoregressive coefficients $\hat{\rho}_{jk,t}$, ($j, k = 1, 2,\ t = 1, ..., T$) plotted with the red line. The left panels show the sample auto- or cross-correlations of r_t^S

and tv_t^S at lag one. In all panels, the green and blue lines are indicating the upper and the lower bounds of confidence intervals, respectively.

Figure 6 compares the dynamics of $AC(r^S)_t$ and the time series $\hat{\rho}_{11,t}$. We observe that the auto-correlations of SPDR returns increase over time and have two major troughs in October 2008 and October 2011. After year 2012, the auto-correlations remain steady and positive. We observe similar dynamics, although at a different level in $\hat{\rho}_{11,t}$ in the right hand side of Figure 6. The series $\hat{\rho}_{11,t}$ is always negative, and it is growing from December 2008 until August 2012. After August 2012, it starts to decrease. Before August 2012, it has two major peaks on January 2009 and August 2012 and two major drops on March 2010 and October 2011. After August 2012 the series $\hat{\rho}_{11,t}$ reaches its lowest value on July 2015.

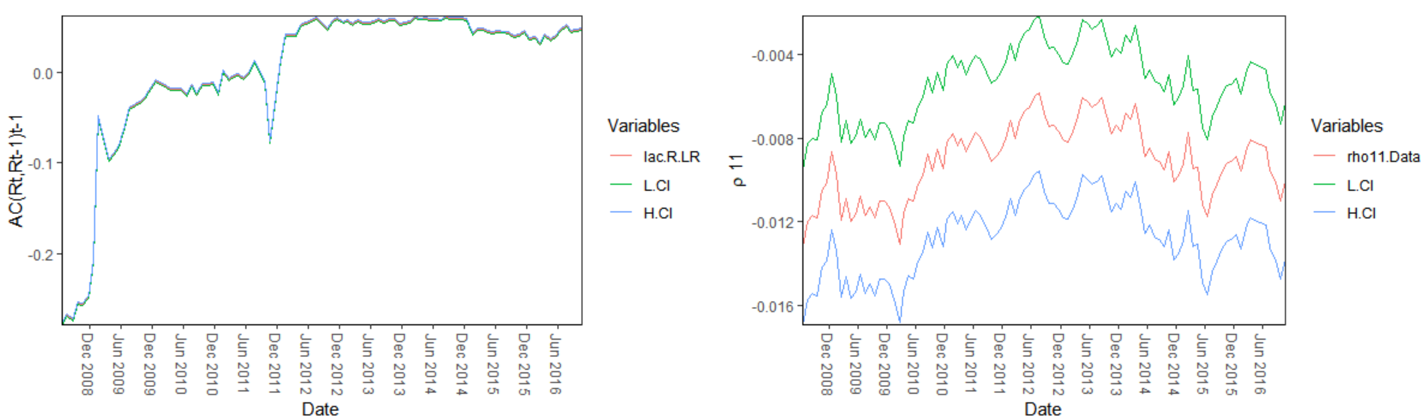

Figure 6: Time Series of SPDR Auto-correlations $AC(r^S)_t$ and Coefficients $\hat{\rho}_{11t}$

Figure 6 compares the sample auto-correlations at lag one of SPDR's returns with the time series of autoregressive coefficients $\hat{\rho}_{11t}$ from model (2.10). The red line in the left plot shows the sample auto-correlations of (r_t^S, r_{t-1}^S) and the coefficients $\hat{\rho}_{11t}$ in the right plot. In both plots the green and blue lines show the upper and lower bounds of confidence intervals.

Figure 7 shows the dynamics of $\text{CC}(r^S, tv^S)_t$ compared to the time series $\hat{\rho}_{12t}$ in the right plot. Both the cross-correlations and $\hat{\rho}_{12t}$ are decreasing over time. In the left plot, the cross-correlations between r_t and tv_{t-1} have two peaks on October 2011 and 2015. In the right plot, we observe that the series $\hat{\rho}_{12t}$ has three major peaks on February 2010, June 2012 and July 2016. Figure 8 compares the dynamics of sample cross-correlations $CC(tv^S, r^S)_t$ of SPRD with the time series of coefficients $\hat{\rho}_{21t}$. The cross-correlations decrease until February 2010 and increase afterwards.

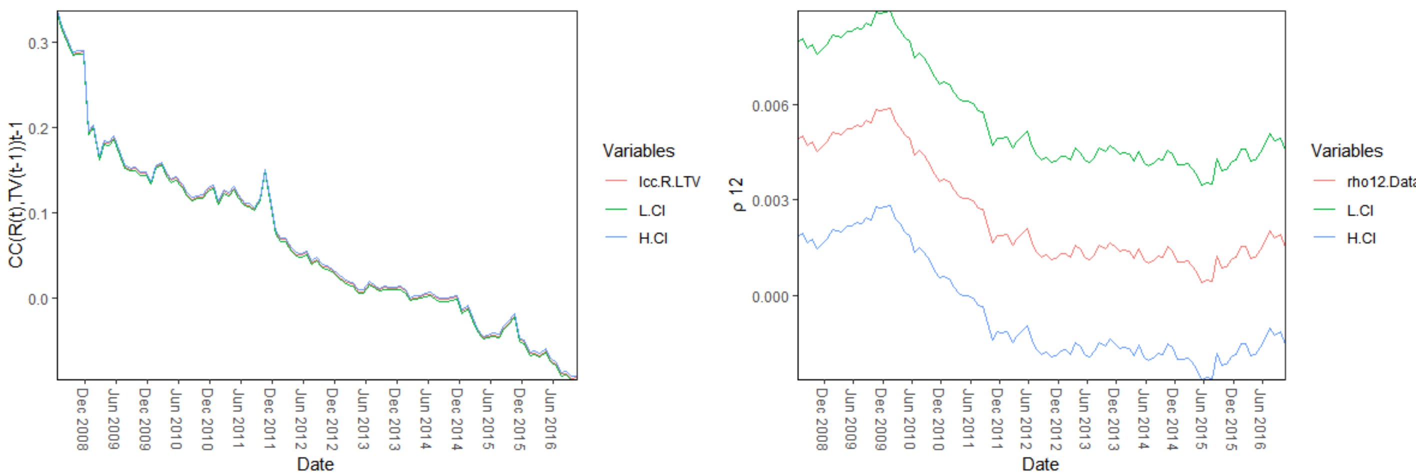

Figure 7: Time Series of SPDR Cross-correlations $CC(r^S, tv^S)_t$ and Coefficients $\hat{\rho}_{12t}$

Figure 7 compares the sample cross-correlations of (r_t^S, tv_{t-1}^S) of SPDR and the time series of coefficients $\hat{\rho}_{12t}$ from model (2.10). The red line in the left plot shows the sample cross-correlations of (r_t^S, tv_{t-1}^S) and the coefficients $\hat{\rho}_{12t}$ in the right plot. In both plots the green and blue lines show the upper and lower bounds of confidence intervals.

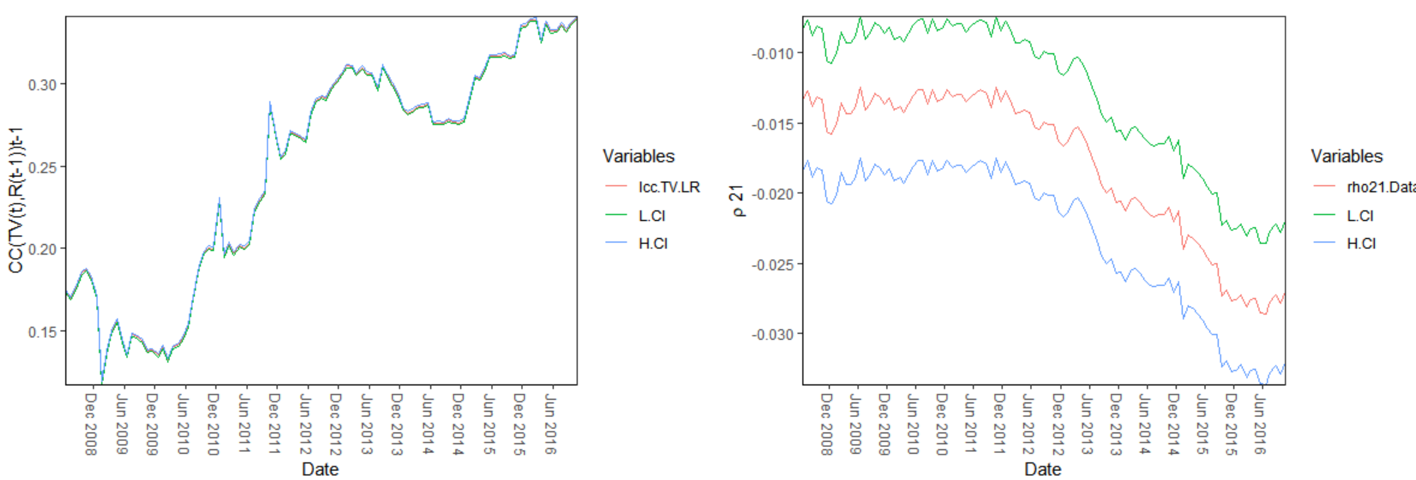

Figure 8: Time Series of SPDR Cross-correlations $CC(tv^S, r^S)_t$ and Coefficients $\hat{\rho}_{21t}$

Figure 8 compares the sample cross-correlations of (tv_t^S, r_{t-1}^S) of SPDR with coefficients $\hat{\rho}_{21t}$ from model (2.10). The red line in the left plot shows the sample cross-correlations of (tv_t^S, r_{t-1}^S) and the coefficients $\hat{\rho}_{21t}$ in the right plot. In both plots the green and the blue lines show the upper and lower bounds of confidence intervals.

They reach the minimum value on December 2008 while always remaining positive. The series of coefficients, $\hat{\rho}_{21t}$ always takes negative values. Coefficients $\hat{\rho}_{21t}$ stay at a constant level until December 2011, and decrease afterwards.

Figure 9, shows the sample auto-correlations $\mathrm{AC}(tv^S)_t$ and the time series of coefficients $\hat{\rho}_{22t}$. The dynamics of these two series are different, but they both always take negative values. The auto-correlations at lag one of trade volume changes of SPDR reach their first peak on January 2009 and drop to their minimum value on September 2011. Next, that series grows until November 2014 and then drops to its second minimum value on October 2015. In the right plot, we observe that series $\hat{\rho}_{22t}$ increases until April 2012, and decreases afterwards.

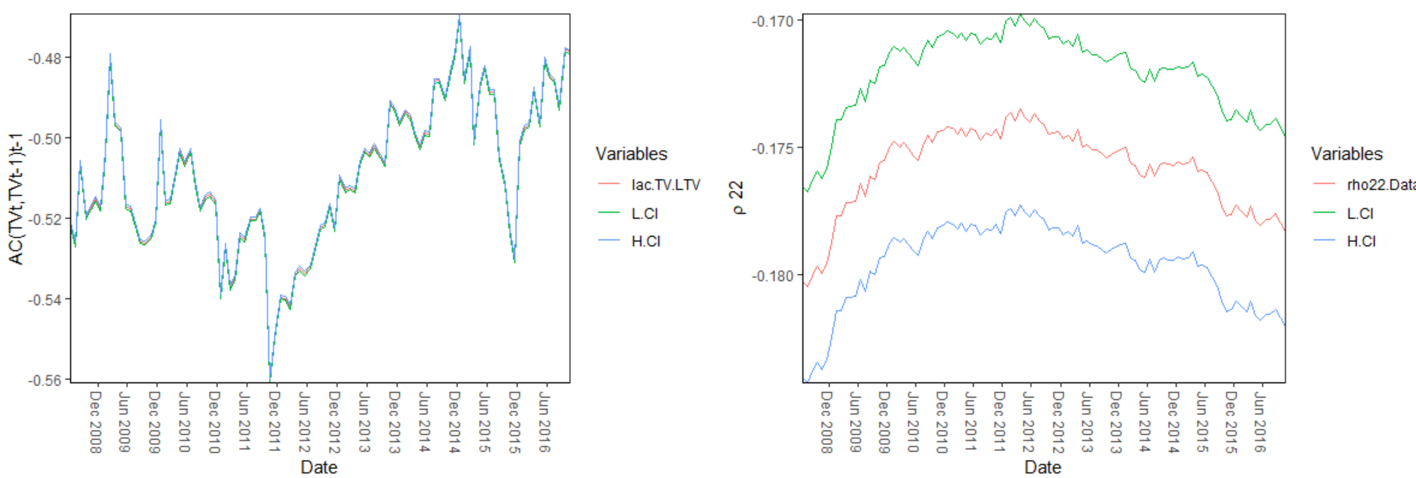

Figure 9: Time Series of Auto-correlations $AC(tv^S)_t$ and Coefficients $\hat{\rho}_{22t}$

Figure 9 compares the sample auto-correlations at lag one of SPDR's trade volume changes with the time series of coefficients $\hat{\rho}_{22t}$ from model (2.10). The red line in the left plot shows the sample auto-correlations of (tv_t^S, tv_{t-1}^S) and the coefficients $\hat{\rho}_{22t}$ in the right plot. In both plots the green and the blue lines show the upper and lower bounds of confidence intervals.

The empirical analysis of the dynamics and distributional properties of autoregressive coefficients $\hat{\rho}_{jk,t}$ and sample auto- and cross-correlations of SPDR returns and trade volume changes leads to the modelling of autoregressive coefficients as functions of the sample correlation functions of SPDR.

3.4 Dynamic Factor Models of $\rho_{jk,t}$

The following regressions reveal the existence of statistically significant linear relationship between the series of autoregressive coefficients $\rho_{jkt},\ (jk = 1, 2,\ t = 1, ..., T)$ and the auto- and cross-correlations of SPDR's return and trade volume changes.

$$\hat{\rho}_{11,t} = a_{110} + a_{11}AC(r^S)_{t-1} + d_{1,t}, \tag{3.12}$$

$$\hat{\rho}_{12,t} = a_{120} + a_{12}CC(r^S, tv^S)_{t-1} + d_{2,t}, \tag{3.13}$$

$$\hat{\rho}_{21,t} = a_{210} + a_{21}CC(tv^S r^S)_{t-1} + d_{3,t}, \tag{3.14}$$

$$\hat{\rho}_{22,t} = a_{220} + a_{22}AC(tv^S)_{t-1} + d_{4,t}, \tag{3.15}$$

where $\hat{\rho}_{11,t}, \hat{\rho}_{12,t}, \hat{\rho}_{21,t}$ and $\hat{\rho}_{22,t}$ are the series of autoregressive coefficients of VAR(1) model (2.10) displayed in Figures 1 and 2, and $AC(r^S)_{t-1}$ and $AC(tv^S)_{t-1}$ are the lagged values of auto-correlations of SPDR return and trade volume changes, $CC(r^S tv^S)_{t-1}$ is the lagged value of the cross-correlation between r_t^S and tv_{t-1}^S, $CC(tv^S r^S)_{t-1}$ is the lagged value of the cross-correlation between tv_t^S and r_{t-1}^S. Parameters $a_{110}, a_{120}, a_{210}$ and a_{220} are the intercepts, a_{11}, a_{12}, a_{21} and a_{22} are the regression coefficients and $d_{1,t}, d_{2,t}, d_{3,t}$ and $d_{4,t}$ are the disturbance terms which are assumed to have mean zero, fixed variances and are orthogonal to the regresses. Table 5 shows the results of estimating the above linear regressions:

Table 5: Linear Regression Coefficients

Dependent Variable	a_{jk0}	a_{jk}	R^2	RSE
ρ_{11}	-0.009***	0.012***	0.35	0.001
ρ_{12}	0.001***	0.014***	0.68	0.001
ρ_{21}	-0.003*	-0.058***	0.58	0.003
ρ_{22}	-0.192***	-0.031***	0.12	0.002

$^{***}p < 0.01$, $^{**}p < 0.05$, $^{*}p < 0.1$

Note: Table 5 shows the results of estimating linear equations (3.12)-(3.15). a_{jk0} show the intercepts, a_{jk} show the regression coefficients, R^2 shows the multiple R-squared and RES shows the residual standard error.

All regression coefficients are statistically significant [11]. This result implies that by using the lagged values of auto- and cross-correlations of SPDR's return and trade volume changes, we can predict the parameters of the VAR(1) model as follows:

$$\hat{\hat{\rho}}_{11,t} = \hat{a}_{110} + \hat{a}_{11} AC(r^S)_{t-1}, \tag{3.16}$$

$$\hat{\hat{\rho}}_{12,t} = \hat{a}_{120} + \hat{a}_{12} CC(r^S, tv^S)_{t-1}, \tag{3.17}$$

$$\hat{\hat{\rho}}_{21,t} = \hat{a}_{210} + \hat{a}_{21} CC(tv^S r^S)_{t-1}, \tag{3.18}$$

$$\hat{\hat{\rho}}_{22,t} = \hat{a}_{220} + \hat{a}_{22} AC(tv^S)_{t-1}. \tag{3.19}$$

Next, the fitted values of $\hat{\rho}_{11}, \hat{\rho}_{12}, \hat{\rho}_{21}$ and $\hat{\rho}_{22}$ are computed from equations (3.16) to (3-19). The following figures show the fitted series $\hat{\hat{\rho}}_{11t}, \hat{\hat{\rho}}_{12t}, \hat{\hat{\rho}}_{21t}$ and $\hat{\hat{\rho}}_{22t}$ and compare them to the dependent variables $\hat{\rho}_{11t}, \hat{\rho}_{12t}, \hat{\rho}_{21t}$ and $\hat{\rho}_{22t}$.

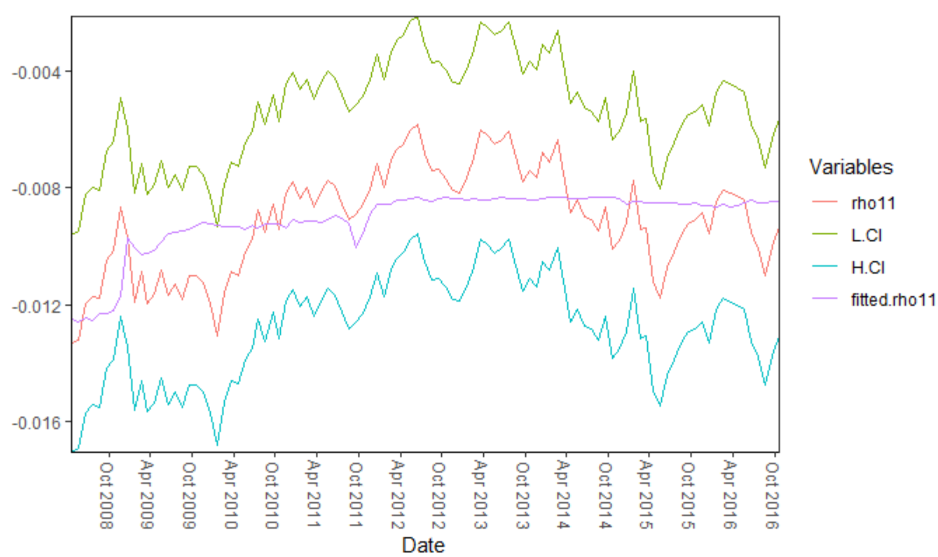

Figure 10: Time Series of $\hat{\rho}_{11t}$ and Fitted Values $\hat{\hat{\rho}}_{11t}$

Figure 10 compares the time series of estimated $\hat{\rho}_{11t}$ and the fitted values of $\hat{\hat{\rho}}_{11t}$. The red line shows the estimated $\hat{\rho}_{11t}$ from VAR(1) model (2.10), green and blue lines show it's upper and lower confidence intervals. The purple line shows the fitted values of $\hat{\hat{\rho}}_{11t}$.

[11]The regression lines are provided in Appendix B.

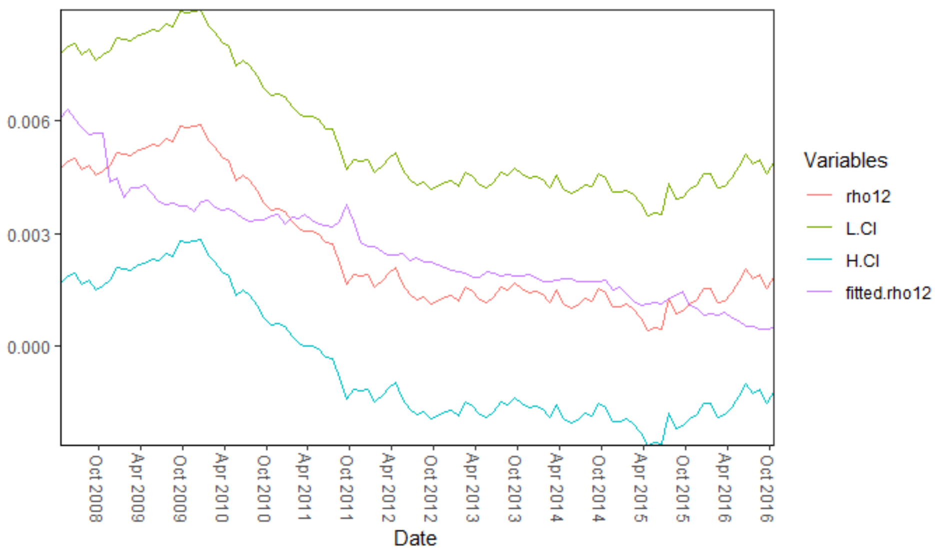

Figure 11: Time Series of $\hat{\rho}_{12t}$ and Fitted Values $\hat{\hat{\rho}}_{12t}$

Figure 11 compares the time series of estimated $\hat{\rho}_{12t}$ and the fitted values $\hat{\hat{\rho}}_{12t}$. The red line shows the estimated $\hat{\rho}_{12t}$ from VAR(1) model (2.10), green and blue lines show it's upper and lower confidence intervals. The purple line shows the fitted values of $\hat{\hat{\rho}}_{12t}$.

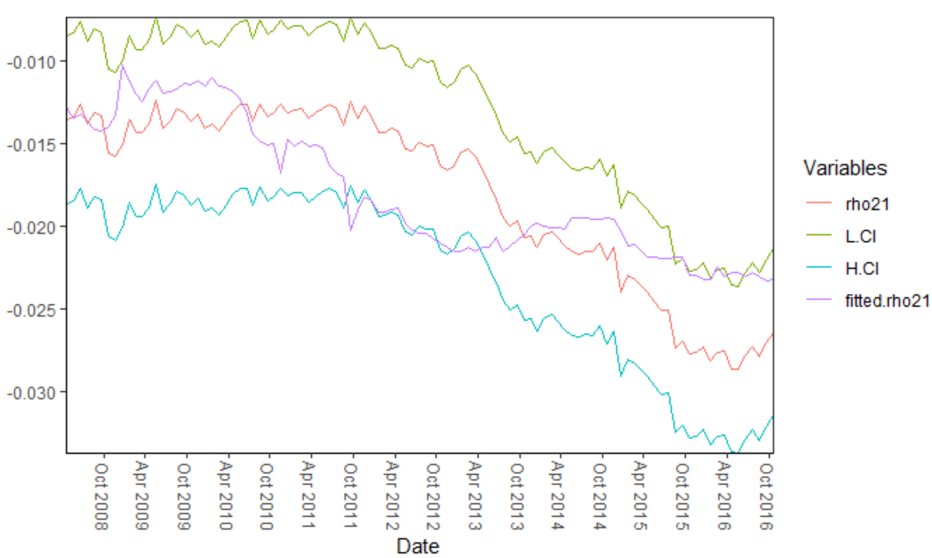

Figure 12: Time Series of $\hat{\rho}_{21t}$ and Fitted Values $\hat{\hat{\rho}}_{21t}$

Figure 12 compares the time series of estimated $\hat{\rho}_{21t}$ and the fitted values $\hat{\hat{\rho}}_{21t}$. The red line shows the estimated $\hat{\rho}_{21t}$ from VAR(1) model (2.10), green and blue lines show it's upper and lower confidence intervals. The purple line shows the fitted values of $\hat{\hat{\rho}}_{21t}$.

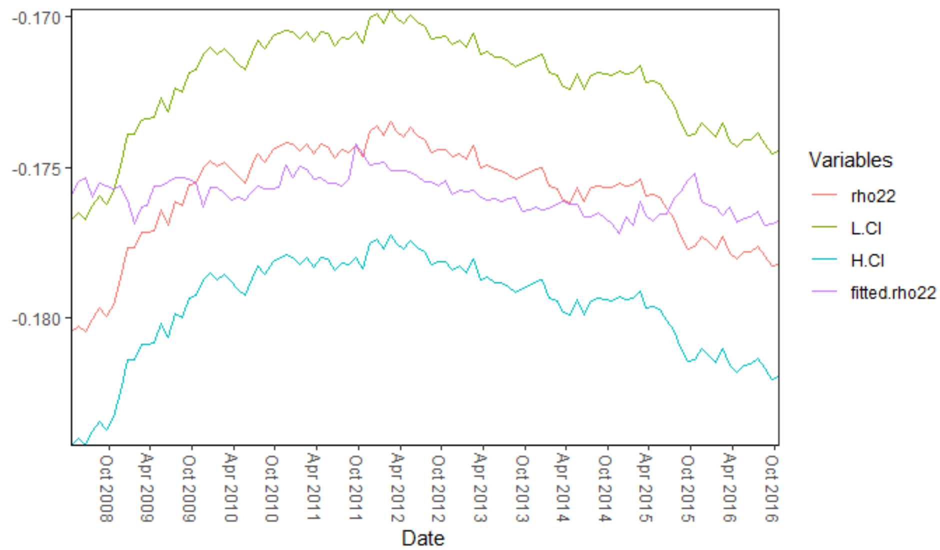

Figure 13: Time Series of $\hat{\rho}_{22t}$ and Fitted Values $\hat{\hat{\rho}}_{22t}$

Figure 13 compares the time series of estimated $\hat{\rho}_{22t}$ and the fitted values $\hat{\hat{\rho}}_{22t}$. The red line shows the estimated $\hat{\rho}_{22t}$ from VAR(1) model (2.10), green and blue lines show it's upper and lower confidence intervals. The purple line shows the fitted values of $\hat{\hat{\rho}}_{22t}$.

In all four plots, the fitted values of autoregressive coefficients show less fluctuation then the estimates. However, their patterns are close to those of the the estimated autoregressive parameters and they remain inside the confidence intervals of the estimated autoregressive parameters. As we can see in all these Figures, at the end of the sample period there is a gap between the fitted value and the time series of the coefficients. To reduce the gap between the estimated ($\hat{\rho}_{jk}$) and fitted coefficients ($\hat{\hat{\rho}}_{jk}$) at the end of the sampling period, for out-of-sample forecasts, the fit can be adjusted locally, by calibrating the regression coefficients.

3.5 Rank Forecasts

The previous Section showed that the $SPDR$ approximates the behavior of the market portfolio as it returns are close to those of $S\&P500$. Therefore, it can be considered as an observable factor. It follows that the four explanatory variables in equations (3.16) to (3.19) can be considered as fixed functions of factor returns and trade volume changes, determining the autoregressive coefficients of the VAR(1) model and the persistence of stock return and liquidity ranks.

This result provides an alternative approach to forecasting out of sample the future ranks of stock returns and volume changes from the VAR(1) model (2-10). At time $T+1$, the future true rank is:

$$\begin{pmatrix} u_{iT+1} \\ v_{iT+1} \end{pmatrix} = \begin{pmatrix} \rho_{11,T+1} & \rho_{12,T+1} \\ \rho_{21,T+1} & \rho_{22,T+1} \end{pmatrix} \begin{pmatrix} u_{i,T} \\ v_{i,T} \end{pmatrix} + \begin{pmatrix} \epsilon_{1,iT+1} \\ \epsilon_{2,iT+1} \end{pmatrix}, \qquad i = 1, \cdots, n. \tag{3.20}$$

It can be forecast using the last values of coefficients $\rho_{jkT},\ j,k = 1,2$ estimated by rolling and displayed in Figures 1 and 2. This approach assumes implicitly that the autoregressive coefficients remain constant between times T and $T+1$. Then the estimated ranks is as follows:

$$\begin{pmatrix} \hat{u}_{iT+1} \\ \hat{v}_{iT+1} \end{pmatrix} = \begin{pmatrix} \hat{\rho}_{11,T} & \hat{\rho}_{12,T} \\ \hat{\rho}_{21,T} & \hat{\rho}_{22,T} \end{pmatrix} \begin{pmatrix} u_{i,T} \\ v_{i,T} \end{pmatrix}, \qquad i = 1, \cdots, n, \tag{3.21}$$

Instead of re-estimating the VAR(1) by rolling equation 2.10, one can find the autoregressive coefficients by computing the fitted values $\hat{\hat{\rho}}_{jk,T+1}$ from equations (3.16) to (3.19) and by using the fixed values of linear regression parameters given in Table 5 as follows:

$$\begin{pmatrix} \hat{u}_{iT+1} \\ \hat{v}_{iT+1} \end{pmatrix} = \begin{pmatrix} \hat{\hat{\rho}}_{11,T+1} & \hat{\hat{\rho}}_{12,T+1} \\ \hat{\hat{\rho}}_{21,T+1} & \hat{\hat{\rho}}_{22,T+1} \end{pmatrix} \begin{pmatrix} u_{i,T} \\ v_{i,T} \end{pmatrix}, \qquad i = 1, \cdots, n, \tag{3.22}$$

The relative performance of the two forecast methods is assessed empirically in Section 5. In the next Section, the predicted ranks of returns and trade volumes are used as the approximations of the expected future ranks to build optimal portfolio allocations.

4 Optimal Positional Management

In this Section we determine the optimal portfolio allocations for an investor with a CARA utility function.

4.1 Optimal Positional Allocations

The empirical analysis presented in the previous Section concerned the empirical ranks of returns and trade volume changes transformed to Gaussian variables. The portfolio management, however is based on their theoretical counterparts. Therefore, we distinguish and define the theoretical ex-ante ranks from the assumed theoretical c.d.f of each of these two series, denoted by F_t^r and F_t^{tv}. Then, the ex-ante ranks are defined as follows:

$$u_{it}^* = F_t^r(r_{it}), \tag{4.23}$$

$$v_{it}^* = F_t^{tv}(tv_{it}), \tag{4.24}$$

The theoretical Gaussian ranks are given by $u_{it} = \Phi^{-1}(u_{it}^*) = Q_t^r(r_{it})$ and $v_{it} = \Phi^{-1}(v_{it}^*) = Q_t^{tv}(tv_{it})$, where $Q_t^r = \Phi^{-1} \circ F_t^r$ and $Q_t^{tv} = \Phi^{-1} \circ F_t^{tv}$. As the Gaussian ranks of returns and volume changes are Normally cross-sectionally distributed, at each time t the relationship between asset i returns and trade volume changes and their respective ranks can be defined by the following stochastic transformations:

$$r_{i,t} = \sigma_{r,t} u_{it} + \mu_{r,t} \qquad t = 1, \cdots, T \;\; , i = 1, \cdots, n, \tag{4.25}$$

$$tv_{i,t} = \sigma_{tv,t} v_{it} + \mu_{tv,t} \qquad t = 1, \cdots, T \;\; , i = 1, \cdots, n, \tag{4.26}$$

where $\mu_{r,t}, \mu_{tv,t}$ are the cross-sectional means of returns and trade volume changes and $\sigma_{r,t}, \sigma_{tv,t}$, represent the cross-sectional standard deviations of the marginal Normal distributions of return and trade volume changes at time t. This transformation, implies that the cross-sectional marginal distributions of assets' returns and trade volume changes at date t are Gaussian as well ($N(\mu_{r,t}, \sigma_{r,t})$ and $N(\mu_{tv,t}, \sigma_{tv,t})$ respectively).

Let us consider two types of investors; investor 1 is looking for a portfolio that provides the highest possible future return rank and investor 2 is looking for a portfolio with the highest possible future liquidity rank.

The quantile functions are time varying and are given below for the return and trade volume changes, respectively:

$$Q_t^r(r_{it}) = \frac{r_{it} - \mu_{r,t}}{\sigma_{r,t}} \tag{4.27}$$

$$Q_t^{tv}(tv_{it}) = \frac{tv_{it} - \mu_{tv,t}}{\sigma_{tv,t}} \tag{4.28}$$

The investor maximizes a CARA utility function in either return or trade volume changes subject to a constraint $\beta' h = 1$, where h is a unit vector of length n. This implies that the sum of all portfolio allocations is equal to one, and the optimal portfolio contains risky assets only. For investors 1 and 2, the future ranks of portfolio returns and traded volumes are $Q_{t+1}^r(\beta_r' r_{t+1})$ and $Q_{t+1}^{tv}(\beta_{tv}' tv_{t+1})$, respectively. These investors maximize the conditional expected utilities $E_t \mathscr{U}[Q_{t+1}^r(\beta_r' r_{t+1})]$ and $E_t \mathscr{U}[Q_{t+1}^{tv}(\beta_{tv}' tv_{t+1})]$, where $\mathscr{U}(u) = -exp(-\mathscr{A}_r u)$ or $\mathscr{U}(v) = -exp(-\mathscr{A}_{tv} v)$ and E_t denotes the expectation conditional on the current and past returns, volumes and the predetermined current values of the autoregressive coefficients. Therefore, the optimal positional momentum strategy consists in selecting assets with the optimal relative allocation vector $\hat{\beta}_{r,t}$, where:

$$\begin{aligned} \beta_{r,t}^* &= \underset{\beta_r : \beta_r' h = 1}{\arg\max} \; E_t[\mathscr{U}(Q_{t+1}^r(\beta_r' r_{t+1}] \\ &= \underset{\beta_r : \beta_r' h = 1}{\arg\max} \; E_t[\mathscr{U}(Q_{t+1}^r\Big(\sum_{i=1}^n \beta_{r,i} {Q_{t+1}^r}^{-1}(u_{it+1})\Big))], \end{aligned} \tag{4.29}$$

The optimal positional allocation vector based on the liquidity ranks is:

$$\begin{aligned} \beta_{tv,t}^* &= \underset{\beta_{tv} : \beta_{tv}' h = 1}{\arg\max} \; E_t[\mathscr{U}(Q_{t+1}^{tv}(\beta_{tv}' tv_{t+1}] \\ &= \underset{\beta_{tv} : \beta_{tv}' h = 1}{\arg\max} \; E_t[\mathscr{U}(Q_{t+1}^{tv}\Big(\sum_{i=1}^n \beta_{tv,i} {Q_{t+1}^{tv}}^{-1}(v_{it+1})\Big))] \end{aligned} \tag{4.30}$$

Let us consider the positional momentum and liquid portfolios which each contains relative risky allocation vectors β_r' and β_{tv}', respectively. The future return and trade volume change of these portfolios are given by:

$$\beta_r' r_{t+1} = \sigma_{r,t+1} \beta_r' u_{t+1} + \mu_{r,t+1} \beta_r' h \tag{4.31}$$

$$\beta_{tv}' tv_{t+1} = \sigma_{tv,t+1}\beta_{tv}' v_{t+1} + \mu_{tv,t+1}\beta_{tv}' h \tag{4.32}$$

since $\beta_r' h = \beta_{tv}' h = 1$. By substituting the future positions of return and volume change ranks into (4.27) and (4.28) respectively, the future positions of the portfolios become:

$$Q_{t+1}^{r}(\beta_r' r_{t+1}) = \frac{\sigma_{r,t+1}\beta_r' u_{t+1} + \mu_{r,t+1}\beta_r' h - \mu_{r,t+1}}{\sigma_{r,t+1}} = \beta_r' u_{t+1}, \qquad \forall \ \beta_r, \tag{4.33}$$

$$Q_{t+1}^{tv}(\beta_{tv}' tv_{t+1}) = \frac{\sigma_{tv,t+1}\beta_{tv}' v_{t+1} + \mu_{tv,t+1}\beta_{tv}' h - \mu_{tv,t+1}}{\sigma_{tv,t+1}} = \beta_{tv}' v_{t+1}, \qquad \forall \ \beta_{tv}, \tag{4.34}$$

Equations (4.33) and (4.34) show that, the position of the future return and trade volume change of the momentum and liquid positional portfolios is a linear combination of the future Gaussian ranks of return and trade volume changes of the individual risky asset $(u_{i,t+1}, v_{i,t+1})$, with weights equal to the elements of the relative risky allocations β_r and β_{tv}. The future positions of the return and trade volume of the portfolios are equal to the shares of each asset in the portfolio multiplied by its future rank. Therefore, in order to predict the future positions of returns and trade volumes of the portfolio, we can use their future Gaussian ranks weighted by their respective shares in each portfolio. This result is a consequence of the linearity of the transformed quantile function (Q_{t+1}) under the Normality assumption on the cross-sectional distributions [see equations (4.27)-(4.28)], and holds for any dynamics of the ranks.

More specifically, by considering the dynamics of ranks introduced in the panel VAR model (equation 2.10), the future positions of returns and trade volume changes can be written as functions of their current ranks as follows:

$$\begin{aligned} Q_{t+1}^{r}(\beta_r' r_{t+1}) &= \beta_r' u_{t+1} \\ &= \sum_{i=1}^{n} \beta_{r,i}\rho_{11,t+1} u_{i,t} + \sum_{i=1}^{n} \beta_{r,i}\rho_{12,t+1} v_{i,t} + \sum_{i=1}^{n} \beta_{r,i}\epsilon_{1,it+1} \end{aligned} \tag{4.35}$$

$$\begin{aligned} Q_{t+1}^{tv}(\beta_{tv}' tv_{t+1}) &= \beta_{tv}' v_{t+1} \\ &= \sum_{i=1}^{n} \beta_{tv,i}\rho_{21,t+1} u_{i,t} + \sum_{i=1}^{n} \beta_{tv,i}\rho_{22,t+1} v_{i,t} + \sum_{i=1}^{n} \beta_{tv,i}\epsilon_{2,it+1}, \end{aligned} \tag{4.36}$$

where coefficients $\rho_{11,t+1}, \rho_{12,t+1}, \rho_{21,t+1}$ and $\rho_{22,t+1}$, in the autoregressive matrix R_t and the conditional variance matrix Σ_t are assumed to be predetermined and are known to the investor at time t [12]. These equations show that the future positions of returns and trade volume changes can be easily computed from the current ranks of returns and trade volume changes.

In the optimizations (4.29) and (4.30), the risk aversion coefficients $\mathscr{A}$ depends on the investor. We assume that the risk aversion of Investor 1 is $\mathscr{A}_r$ and that of investor 2 is $\mathscr{A}_{tv}$. After substituting the quantile functions in the utility function and given that errors $\epsilon_{1,it}, \epsilon_{2,it}$ in equation (210), are independent Gaussian white noise processes, the expected positional utilities to be maximized are as follows:

$$\begin{gathered} -E[exp(-\mathscr{A}_r Q^r_{t+1}(\beta'_r r_{t+1})) \mid \underline{r_t}, \underline{tv_t}, R_{t+1}] = \\ -\Big[exp\Big(-\mathscr{A}_r \sum_{i=1}^{n} \beta_{r,i}\rho_{11,t+1}u_{i,t} - \mathscr{A}_r \sum_{i=1}^{n} \beta_{r,i}\rho_{12,t+1}v_{i,t} + \frac{1}{2}\mathscr{A}_r^2 \sum_{i=1}^{n} \beta^2_{r,i}\sigma^2_{1,t+1}\Big)\Big] \end{gathered} \tag{4.37}$$

where $\sigma^2_{1t+1} = 1 - \rho^2_{11,t+1} - \rho^2_{12,t+1}$, subject to $\beta'_r h = 1$ and,

$$\begin{gathered} -E[exp(-\mathscr{A}_{tv} Q^{tv}_{t+1}(\beta'_{tv} tv_{t+1})) \mid \underline{r_t}, \underline{tv_t}, R_{t+1}] = \\ -\Big[exp\Big(-\mathscr{A}_{tv} \sum_{i=1}^{n} \beta_{tv,i}\rho_{21,t+1}u_{i,t} - \mathscr{A}_{tv} \sum_{i=1}^{n} \beta_{tv,i}\rho_{22,t+1}v_{i,t} + \frac{1}{2}\mathscr{A}_{tv}^2 \sum_{i=1}^{n} \beta^2_{tv,i}\sigma^2_{2,t+1}\Big)\Big] \end{gathered} \tag{4.38}$$

where $\sigma^2_{2t+1} = 1 - \rho^2_{21,t+1} - \rho^2_{22,t+1}$, subject to $\beta'_{tv} h = 1$, for investor 2. In each of the above equations ((4.37) and (4.38)), the expected positional utility is independent of the cross-sectional mean and standard deviation of returns and trade volumes ($\mu_{r,t}, \mu_{tv,t}$ and $\sigma_{r,t}, \sigma_{tv,t}$) at time t and depends on the current position of asset i return and trade volume change (u_{it} and v_{it}).

The Lagrangian functions for the maximization of the expected positional utility with respect to the portfolio allocation vectors β'_r and β'_{tv}, subject to the constraints $\beta'_r h = 1$

[12] See Appendix C for a heuristic demonstration of case of stochastic autoregressive coefficients.

and $\beta_{tv}'h = 1$ are:

$$\begin{aligned} L_r = -\Big[exp\Big(- \mathscr{A}_r \sum_{i=1}^{n} \beta_{r,i}\rho_{11,t+1}u_{i,t} - \mathscr{A}_r \sum_{i=1}^{n} \beta_{r,i}\rho_{12,t+1}v_{i,t} + \\ \frac{1}{2}\mathscr{A}_r^2 \sum_{i=1}^{n} \beta_{r,i}^2\sigma_{1,t+1}^2\Big)\Big] + \lambda_r(1 - \beta_r'h) \end{aligned} \tag{4.39}$$

$$\begin{aligned} L_{tv} = -\Big[exp\Big(- \mathscr{A}_{tv} \sum_{i=1}^{n} \beta_{tv,i}\rho_{21,t+1}u_{i,t} - \mathscr{A}_{tv} \sum_{i=1}^{n} \beta_{tv,i}\rho_{22,t+1}v_{i,t} + \\ \frac{1}{2}\mathscr{A}_{tv}^2 \sum_{i=1}^{n} \beta_{tv,i}^2\sigma_{2,t+1}^2\Big)\Big] + \lambda_{tv}(1 - \beta_{tv}'h) \end{aligned} \tag{4.40}$$

where λ_r and λ_{tv} are the Lagrange multipliers. The first-order condition for $\beta_{r,t}, \beta_{tv,t}$ are:

$$\begin{aligned} & - \mathscr{A}_r\Big[(\rho_{11,t+1}u_t + \rho_{12,t+1}v_t - \mathscr{A}_r\sigma_{1,t+1}^2\beta_{r,t}) \\ & \qquad exp\Big[- \mathscr{A}_r(\rho_{11,t+1}\beta_{r,t}'u_t - \rho_{12,t+1}\beta_{r,t}v_t) + \frac{1}{2}\mathscr{A}_r^2\beta_{r,t}'\beta_{r,t}\sigma_{1,t+1}^2\Big)\Big] - \lambda_{r,t}h = 0 \end{aligned} \tag{4.41}$$

$$\begin{aligned} & - \mathscr{A}_{tv}\Big[(\rho_{21,t+1}u_t + \rho_{22,t+1}v_t - \mathscr{A}_{tv}\sigma_{2,t+1}^2\beta_{tv}) \\ & exp\Big[- \mathscr{A}_{tv}(\rho_{21,t+1}\beta_{tv,t}'u_t - \rho_{22,t+1}\beta_{tv,t}v_t) + \frac{1}{2}\mathscr{A}_{tv}^2\beta_{tv,t}'\beta_{tv,t}\sigma_{2,t+1}^2\Big)\Big] - \lambda_{tv,t}h = 0 \end{aligned} \tag{4.42}$$

By solving the above equations with respect to $\beta_{r,t}, \lambda_{r,t}$ and $\beta_{tv,t}, \lambda_{tv,t}$ the optimal portfolio shares at time t are as follows:

$$\beta_{r,t}^* = \frac{1}{\mathscr{A}_r}\frac{\rho_{11,t+1}u_t + \rho_{12,t+1}v_t}{\sigma_{1,t+1}^2} - \frac{1}{\mathscr{A}_r^2}\frac{\lambda_{r,t}h}{\sigma_{1,t+1}^2} \tag{4.43}$$

$$\beta_{tv,t}^* = \frac{1}{\mathscr{A}_{tv}}\frac{\rho_{21,t+1}u_t + \rho_{22,t+1}v_t}{\sigma_{2,t+1}^2} - \frac{1}{\mathscr{A}_{tv}^2}\frac{\lambda_{tv,t}h}{\sigma_{2,t+1}^2} \tag{4.44}$$

In terms of vector we get:

$$\beta_{r,t}^*{}'h = \frac{(\rho_{11t+1}u_{.t} + \rho_{12t+1}v_{.t})}{\mathscr{A}_r\sigma_{1,t+1}^2} - n\frac{\lambda_{r,t}}{\mathscr{A}_r^2\sigma_{1,t+1}^2} = 1, \tag{4.45}$$

$$\beta^{*}_{tv,t}{}'h = \frac{(\rho_{21t+1}u_{.t} + \rho_{22t+1}v_{.t})}{\mathscr{A}_{tv}\sigma^2_{2,t+1}} - n\frac{\lambda_{tv,t}}{\mathscr{A}^2_{tv}\sigma^2_{2,t+1}} = 1, \tag{4.46}$$

where $u_{.t}$ and $v_{.t}$ are the vectors of ranks of asset $i = 1, \ldots, n$ at time t. Then we have:

$$\frac{\lambda_{r,t}}{\mathscr{A}^2_r\sigma^2_{1,t+1}} = \frac{\rho_{11t+1}\overline{u_t} + \rho_{12t+1}\overline{v_t}}{\mathscr{A}_r\sigma^2_{1,t+1}} - \frac{1}{n}, \tag{4.47}$$

$$\frac{\lambda_{tv,t}}{\mathscr{A}^2_{tv}\sigma^2_{2,t+1}} = \frac{\rho_{21t+1}\overline{u_t} + \rho_{22t+1}\overline{v_t}}{\mathscr{A}_{tv}\sigma^2_{2,t+1}} - \frac{1}{n}, \tag{4.48}$$

where $\overline{u_t} = \frac{1}{n}\sum\limits_{i=1}^{n} u_{it}$ and $\overline{v_t} = \frac{1}{n}\sum\limits_{i=1}^{n} v_{it}$. By substituting the above expressions into (4.42) and (4.43) we get the vectors of optimal allocations as follow:

$$\beta^{*}_{r,t} = \frac{1}{n}h + \frac{\rho_{11t+1}(u_t - \overline{u_t}h) + \rho_{12t+1}(v_t - \overline{v_t}h)}{\mathscr{A}_r\sigma^2_{1,t+1}} \tag{4.49}$$

$$\beta^{*}_{tv,t} = \frac{1}{n}h + \frac{\rho_{21t+1}(u_t - \overline{u_t}h) + \rho_{22t+1}(v_t - \overline{v_t}h)}{\mathscr{A}_{tv}\sigma^2_{2,t+1}} \tag{4.50}$$

The optimal relative positional allocations $\beta'^{*}_{r,t}, \beta'^{*}_{tv,t}$ (equations (4.49) and (4.50)) are linear combinations of two well-known portfolios. The first one is the equally weighted portfolio with weight 1/n for each asset and the second portfolio is an arbitrage portfolio (i.e. zero-cost portfolio) with dynamic allocations proportional to the deviations of the current ranks from their cross-sectional averages. Since these arbitrage portfolios contain the vector of expected future ranks in deviation from their cross-sectional averages $\Big((\rho_{11,t+1}(u_{it} - \overline{u_t}) + \rho_{12,t+1}(v_{it} - \overline{v_t})$ in equation (4.49) and $(\rho_{21,t+1}(u_{it} - \overline{u_t}) + \rho_{22,t+1}(v_{it} - \overline{v_t}))$ in equation (4.50) $\Big)$, it can be interpreted as a momentum portfolio in equation (4.49) and liquid portfolio in equation (4.50). When the sign of the sum of persistence coefficients $\rho_{jk,t} + \rho_{jj,t}$ (where j,k=1,2) is positive, the arbitrage portfolio will be long in assets with large expected deviation of their future ranks from their cross-sectional average, and when the sum of persistence coefficients is negative then, it will be short in assets with small expected deviation of their future ranks from their cross-sectional average.

This interpretation of the arbitrage part of the positional portfolio implies that the optimal positional allocation deviates from the equally weighted portfolio by over-weighting the assets with larger current ranks, when the sum of persistence coefficients is positive and deviates from the equally weighted portfolio by over-weighting the assets with small current ranks, when the sum of persistence coefficients is negative. The weight of the arbitrage portfolio in the optimal risky allocations $\beta^*_{r,it}$ and $\beta^*_{tv,it}$ are positively correlated with the persistence of ranks coefficients ($\rho_{11,t+1}, \rho_{12,t+1}$ in equation (4.49) and $\rho_{21,t+1}, \rho_{22,t+1}$ in equation (4.50)) and negatively correlated with the risk aversion coefficients ($\mathscr{A}_r$ in equation (4.49) and $\mathscr{A}_{tv}$ in equation (4.50)) of the investors.

The optimal allocation vectors $\beta^*_{r,t}, \beta^*_{tv,t}$ that determine the positional portfolio strategies depend on the choice of the positional utility function and on the positional universe of stocks which is used to compute the ranks. Moreover, these optimal allocations of the positional investor are defined by considering functions Q_{t+1} as the exogenous functions, which in this paper, are the quantile functions.

4.2 Optimal Mixed Positional Allocations

Let us consider investment strategy that select assets with the highest return and liquidity ranks. The optimal allocation vector β^* is obtained by maximizing the positional CARA utility function as follows:

$$\begin{aligned} &- E[exp(-(\mathscr{A}_r Q^r_{t+1}(\beta'_r r_{t+1}) + \mathscr{A}_{tv} Q^{tv}_{t+1}(\beta'_{tv} tv_{t+1}))) \mid \underline{r_t}, \underline{tv_t}, R_{t+1}] \\ &= -E\Big[exp\big(- (\mathscr{A}_r \beta' u_{t+1} + \mathscr{A}_{tv} \beta' v_{t+1}) \mid \underline{r_t}, \underline{tv_t}, R_{t+1}\big)\Big] \end{aligned} \tag{4.51}$$

subject to $\beta' h = 1$. By analogy to the previous section, we predict the future ranks u_{t+1} and v_{t+1} from the bivariate VAR(1) model (equation 2.10) with time varying coefficients, which are considered predetermined at time t. Next we maximize:

$$\begin{aligned} - \Big[exp\Big(&- \mathscr{A}_r \sum_{i=1}^{n} \beta_i (\rho_{11,t+1} u_{it} + \rho_{12,t+1} v_{it}) - \mathscr{A}_{tv} \sum_{i=1}^{n} \beta_i (\rho_{21,t+1} u_{it} + \rho_{22,t+1} v_{it}) + \\ &\frac{1}{2} \sum_{i=1}^{n} \beta_i^2 (\mathscr{A}_r^2 \sigma^2_{1,t+1} + \mathscr{A}_{tv}^2 \sigma^2_{2,t+1} + 2\mathscr{A}_r \mathscr{A}_{tv} \sigma^2_{12,t+1})\Big)\Big] \end{aligned} \tag{4.52}$$

where $\sigma_{12,t+1} = -\rho_{11,t+1}\rho_{21,t+1} - \rho_{12,t+1}\rho_{22,t+1}$. To simplify the exposition, let us use the vector notation:

$$-\Big[exp\Big(- \mathscr{A}_r(\rho_{11,t+1}\beta' u_t + \rho_{12,t+1}\beta' v_t) - \mathscr{A}_{tv}(\rho_{21,t+1}\beta' u_t + \rho_{22,t+1}\beta' v_t) + \frac{1}{2}\beta'\beta(\mathscr{A}_r^2\sigma_{1,t+1}^2 + \mathscr{A}_{tv}^2\sigma_{2,t+1}^2 + 2\mathscr{A}_r\mathscr{A}_{tv}\sigma_{12,t+1})\Big)\Big] \tag{4.53}$$

subject to $\beta' h = 1$. The Lagrangian of the constrained maximization is:

$$L_M = -\Big[exp\Big(- \mathscr{A}_r(\rho_{11,t+1}\beta' u_t + \rho_{12,t+1}\beta' v_t) - \mathscr{A}_{tv}\beta'(\rho_{21,t+1}u_t + \rho_{22,t+1}v_t) + \frac{1}{2}\beta\beta'(\mathscr{A}_r^2\sigma_{1,t+1}^2 + \mathscr{A}_{tv}^2\sigma_{2,t+1}^2 + 2\mathscr{A}_r\mathscr{A}_{tv}\sigma_{12,t+1}^2)\Big)\Big] + \lambda(1 - \beta' h) \tag{4.54}$$

where λ is the Lagrange multiplier. The first-order condition for β_t, λ_t is:

$$[\mathscr{A}_r(\rho_{11,t+1}u_t + \rho_{12,t+1}v_t) + \mathscr{A}_{tv}(\rho_{21,t+1}u_t + \rho_{22,t+1}v_t) - (\mathscr{A}_r^2\sigma_{1,t+1}^2 + \mathscr{A}_{tv}^2\sigma_{2,t+1}^2 + 2\mathscr{A}_r\mathscr{A}_{tv}\sigma_{12,t+1})\beta_t]exp\Big(- \mathscr{A}_r(\rho_{11,t+1}\beta' u_t + \rho_{12,t+1}\beta' v_t) - \mathscr{A}_{tv}(\rho_{21,t+1}\beta' u_t + \rho_{22,t+1}\beta' v_t) + \frac{1}{2}\beta_t'\beta_t(\mathscr{A}_r^2\sigma_{1,t+1}^2 + \mathscr{A}_{tv}^2\sigma_{2,t+1}^2 + 2\mathscr{A}_r\mathscr{A}_{tv}\sigma_{12,t+1}^2)\Big) - \lambda_t h = 0 \tag{4.55}$$

which yields:

$$\beta_t^* = \frac{\mathscr{A}_r(\rho_{11,t+1}u_t + \rho_{12,t+1}v_t) + \mathscr{A}_{tv}(\rho_{21,t+1}u_t + \rho_{22,t+1}v_t)}{\mathscr{A}_r^2\sigma_{1,t+1}^2 + \mathscr{A}_{tv}^2\sigma_{2,t+1}^2 + 2\mathscr{A}_r\mathscr{A}_{tv}\sigma_{12,t+1}} - \frac{\lambda_t h}{\mathscr{A}_r^2\sigma_{1,t+1}^2 + \mathscr{A}_{tv}^2\sigma_{2,t+1}^2 + 2\mathscr{A}_r\mathscr{A}_{tv}\sigma_{12,t+1}} \tag{4.56}$$

Let m_t denotes the nominator of the first term: $m_t = \mathscr{A}_r(\rho_{11,t+1}u_t + \rho_{12,t+1}v_t) + \mathscr{A}_{tv}(\rho_{21,t+1}u_t + \rho_{22,t+1}v_t)$, and Δ_t denotes the common denominator: $\Delta_t = \mathscr{A}_r^2\sigma_{1,t+1}^2 + \mathscr{A}_{tv}^2\sigma_{2,t+1}^2 + 2\mathscr{A}_r\mathscr{A}_{tv}\sigma_{12,t+1}$. We can rewrite equation (4.55) as follows:

$$\beta_t^* = \frac{m_t}{\Delta_t} - \frac{\lambda_t h}{\Delta_t}. \tag{4.57}$$

By taking into account the constraint $\beta_t^{*\prime} h = 1$, we get equation (4.57) in terms of vector as below:

$$\begin{aligned}\beta_t^{*\prime} h &= \frac{m_{.t}}{\Delta_t} - \frac{\lambda_t h' h}{\Delta_t} = 1 \\ &= \frac{m_{.t}}{\Delta_t} - \frac{\lambda n}{\Delta_t} = 1,\end{aligned} \tag{4.58}$$

By solving for $\frac{\lambda_t}{\Delta_t}$ we get:

$$\frac{\lambda}{\Delta_t} = \frac{1}{n}\frac{m_{.t}}{\Delta_t} - \frac{1}{n} \tag{4.59}$$

By substituting equation (4.59) into the expression of β^* (equation (4.57)), we get the optimal allocation vector as follows:

$$\begin{aligned}\beta^* &= \frac{m_t}{\Delta_t} - \frac{\frac{1}{n} m_t}{\Delta_t} + \frac{1}{n} \\ \beta^* &= \frac{1}{n} h + \frac{1}{\Delta_t}(m_t - \overline{m_t} h)\end{aligned} \tag{4.60}$$

which is the optimal allocation vector:

$$\beta_t^* = \frac{1}{n} h + \frac{\mathscr{A}_r(\rho_{11,t+1}(u_t - \overline{u}_t) + \rho_{12,t+1}(v_t - \overline{v}_t))}{\mathscr{A}_r^2 \sigma_{1,t+1}^2 + \mathscr{A}_{tv}^2 \sigma_{2,t+1}^2 + 2\mathscr{A}_r \mathscr{A}_{tv} \sigma_{12,t+1}} + \frac{\mathscr{A}_{tv}(\rho_{21,t+1}(u_t - \overline{u}_t) + \rho_{22,t+1}(v_t - \overline{v}_t))}{\mathscr{A}_r^2 \sigma_{1,t+1}^2 + \mathscr{A}_{tv}^2 \sigma_{2,t+1}^2 + 2\mathscr{A}_r \mathscr{A}_{tv} \sigma_{12,t+1}} \tag{4.61}$$

It is easy to see that the above formula simplifies when $\sigma_{12,t} = 0$:

$$\beta_t^* = \frac{1}{n} + \frac{\mathscr{A}_r(m_{r,t} - \overline{m}_{r,t} h) + \mathscr{A}_{tv}(m_{tv,t} - \overline{m}_{tv,t} h)}{\mathscr{A}_r^2 \sigma_{1,t+1}^2 + \mathscr{A}_{tv}^2 \sigma_{2,t+1}^2}, \tag{4.62}$$

where $(m_{r,t} - \overline{m}_{r,t} h) = \rho_{11,t+1}(u_t - \overline{u}_t h) + \rho_{12,t+1}(v_t - \overline{v}_t h)$ and $(m_{tv,t} - \overline{m}_{tv,t} h) = \rho_{21,t+1}(u_t -$

$\overline{u}_t h) + \rho_{22,t+1}(v_t - \overline{v}_t h)$. We see that:

$$\beta_t^* = \frac{1}{n}h + \frac{\mathscr{A}_r^2\sigma_{1t+1}^2\frac{(m_{r,t}-\overline{m}_{r,t}h)}{\mathscr{A}_r\sigma_{1t+1}^2} + \mathscr{A}_{tv}^2\sigma_{2t+1}^2\frac{(m_{tv,t}-\overline{m}_{tv,t}h)}{\mathscr{A}_{tv}\sigma_{2t+1}^2}}{\mathscr{A}_r^2\sigma_{1,t+1}^2 + \mathscr{A}_{tv}^2\sigma_{2,t+1}^2}$$

$$= \frac{1}{n}h + \pi_{r,t}\Big[\frac{(m_{r,t} - \overline{m}_{r,t}h)}{\mathscr{A}_r\sigma_{1t+1}^2}\Big] + \pi_{tv,t}\Big[\frac{(m_{tv,t} - \overline{m}_{tv,t}h)}{\mathscr{A}_{tv}\sigma_{2t+1}^2}\Big], \tag{4.63}$$

where $\pi_{r,t} = \frac{\mathscr{A}_r^2\sigma_{1t+1}^2}{\mathscr{A}_r^2\sigma_{1,t+1}^2+\mathscr{A}_{tv}^2\sigma_{2,t+1}^2}$ and $\pi_{tv,t} = \frac{\mathscr{A}_{tv}^2\sigma_{2t+1}^2}{\mathscr{A}_r^2\sigma_{1,t+1}^2+\mathscr{A}_{tv}^2\sigma_{2,t+1}^2}$. It follows from equation (4.62), that when $\sigma_{12,t+1} = 0$, the optimal mixed positional allocation contains two portfolios. The first one is the equally weighted portfolio with weights $1/n$ and the second one is a weighted average of the positional momentum and positional liquidity allocations.

4.3 Optimal Positional Portfolios

From the optimal positional allocation vectors we define the following three types of optimal positional portfolios:

Definition 1: The efficient positional momentum portfolio is based on the optimal positional allocation $\beta_{r,t}^*$ which maximizes the $CARA$ positional utility function under condition $\beta_r' h = 1$ for positional risk aversion parameters $\mathscr{A}_r$ and a bivariate VAR model component of returns ranks dynamics.

As the liquidity ensures uninterrupted availability of funds, we extend this approach further and introduce a new positional liquid portfolio which is efficient in terms of liquidity as follows;

Definition 2: The efficient positional liquid portfolio is based on the optimal positional allocations $\beta_{tv,t}^*$ which maximizes the $CARA$ positional utility function under constraint $\beta_{tv}' h = 1$ for positional risk aversion parameters $\mathscr{A}_{tv}$ and a bivariate VAR model component of trade volumes' ranks dynamics.

Some investors are interested in maximizing the returns while also looking for quick access to funds as well. The third approach introduced as a new mixed positional portfolio, which is efficient in terms of both return and liquidity.

Definition 3: The efficient positional mixed portfolio is based on the optimal positional allocations β_t^* which maximizes the $CARA$ positional utility function under constraint $\beta' h = 1$ for risk aversion parameters $\mathscr{A}_r, \mathscr{A}_{tv}$ and a bivariate VAR model of return and trade volumes' ranks dynamics.

5 Optimal Positional Strategies

In this Section, we implement the optimal positional strategies defined in Section 4. The positional strategies are applied to an investment universe corresponding to the $n = 1330$ stocks traded in NASDAQ market from 1999 to 2016. The positional risk aversion parameters are considered constant and take values $0.5, 1, 3, 5$. The expected ranks of returns are predicted from the bivariate VAR(1) model (equation 2.7) of ranks of returns and trade volume changes using either the autoregressive parameters $\hat{\rho}_{jk,t}$, $jk = 1, 2$ estimated by rolling (equation), or $\hat{\hat{\rho}}_{jk,t+1}$, $jk = 1, 2$ predicted from the factor model (equations 3.16-3.19). This strategy provides optimal portfolios with monthly adjustments of asset allocations and equal look-back periods of one month over the period 2008 to 2016. The returns on the positional portfolios are compared with the returns on the equal weighted portfolio (EW) that are obtained from rolling with a window of 108 months.

5.1 Optimal Positional Momentum Portfolios

The optimal positional momentum portfolios contain stocks with allocations β_t^r, defined as follows:

$$\beta_{r,t}^* = \frac{1}{n} h + \frac{\rho_{11,t+1}(u_t - \overline{u}_t) + \rho_{12,t+1}(v_t - \overline{v}_t)}{\mathscr{A}_r \sigma_{1,t+1}^2} \tag{5.64}$$

Table 6, shows the average of the time series of optimal positional portfolios' returns and their standard deviations and compares those returns with the equally weighted portfolio's return. Two types of positional momentum portfolios are considered: the first type has the future ranks predicted with estimated $\hat{\rho}_{11,t}$, $\hat{\rho}_{12,t}$ from the VAR(1) model (equation 2.7), and the second type is computed by using the fitted values of estimated coefficients from equations (3.16) and (3.17) $\hat{\hat{\rho}}_{11,t+1},\ \hat{\hat{\rho}}_{12,t+1}$. The positional momentum portfolios are calculated for four different values of risk aversions ($\mathscr{A}_r = 0.5, 1, 3, 5$). We observe that all portfolios provide positive returns, which are higher than the returns on the EW portfolio. When the risk aversion value increases, the return of the optimal positional momentum portfolios decreases which is consistent with the risk-return trade-off in financial literature [13]. Equivalently lower risk aversion tends to increase returns due to higher undertaken risk.

For all values of risk aversion considered, the positional momentum portfolios based on estimated $\hat{\rho}_{11,t},\ \hat{\rho}_{12,t}$ provide higher returns than the portfolios based on the fitted values of $\hat{\hat{\rho}}_{11,t+1},\ \hat{\hat{\rho}}_{12,t+1}$. When the risk aversion values increase, the difference between the returns on the two types of portfolios diminishes to zero for $\mathscr{A}_r = 5$.

Table 6: Summary of Positional Momentum Portfolios' Returns

	Estimated ρ's		Fitted ρ's	
Risk Aversion	**Mean**	**S-D**	**Mean**	**S-D**
$\mathscr{A}_r = 0.5$	**2.198**	1.176	2.192	1.089
$\mathscr{A}_r = 1$	**1.101**	0.581	1.098	0.534
$\mathscr{A}_r = 3$	**0.370**	0.191	0.369	0.171
$\mathscr{A}_r = 5$	**0.223**	0.120	**0.223**	0.105
	Mean		S-D	
EW	0.004		0.067	

Note: Table 6 shows the average of the time series return of the optimal positional momentum portfolios with the future ranks predicted with estimated $\hat{\rho}_{11,t},\ \hat{\rho}_{12,t}$ from VAR(1) model (equation 2.7) (Estimated ρ's) and the with fitted value of estimated coefficients from equations (3.16) and (3.17) $\hat{\hat{\rho}}_{11,t+1},\ \hat{\hat{\rho}}_{12,t+1}$ (Fitted ρ's).

Figure 14 shows the time series of returns on the positional momentum portfolios for different values of risk aversion. Both positional momentum portfolios (based on estimated ρ's and fitted $\rho's$) with risk aversion equal to 0.5 outperform the other portfolios. Among these two types of portfolios the positional momentum portfolio based on fitted ρ's performs better until January 2009 and between July 2010 and March 2014. The positional momentum portfolio based on estimated ρ's provides the highest returns between January

[13]Many literature show that the more return sought, the more risk that must be undertaken (Breen, Glosten,and Jagannathan (1989), Nelson (1991), Glosten, Jakannatha and Runkle (1993), Brandtand Kang (2004), etc).

2009 and June 2010 and after July 2014. The EW portfolio provides the lowest returns.

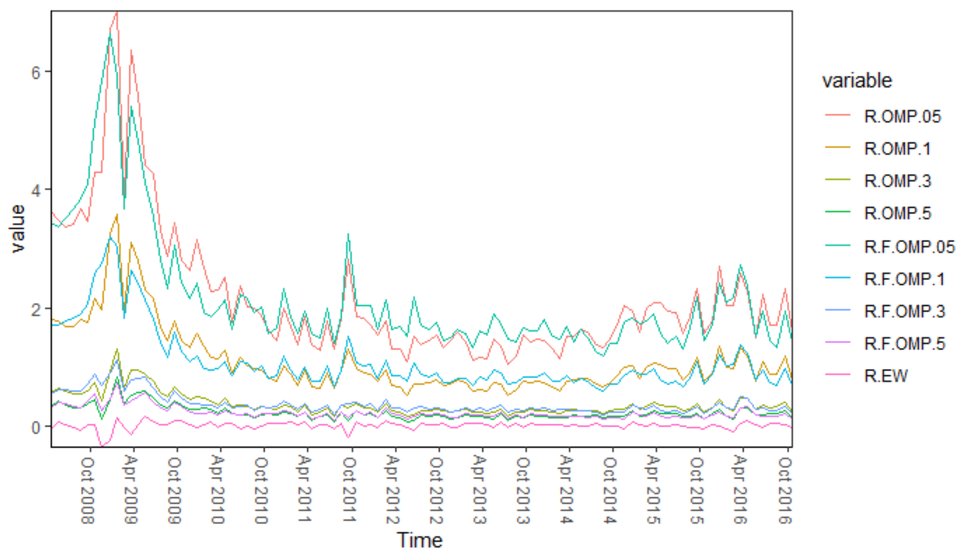

Figure 14: Time Series of Positional Momentum Strategies' Returns

Figure 14 compares the time series of returns of positional momentum portfolios. The red, orange, olive and green line show the returns of optimal positional momentum portfolios computed from estimated parameters of VAR model when $\mathscr{A}_r = 0.5, 1, 3,$ and 5 respectively. The light green, light blue, blue and purple line show the returns of optimal positional momentum portfolios computed from fitted values of parameters from equations (3.16) and (3.17) when $\mathscr{A}_r = 0.5, 1, 3,$ and 5 respectively. The pink line shows the mean of the equally weighted portfolio.

Table 7 shows the cumulative return on the optimal positional momentum portfolios with the inception date of April 2008 until October 2016. For all values of risk aversion considered, the positional momentum portfolios based on estimated ρ's provide higher cumulative return than the portfolios based on fitted ρ's, although these cumulative returns are very close. Figure 15 shows the time series of cumulative returns on all positional momentum portfolios. The positional momentum portfolios (based on estimated and fitted $\rho's$) with risk aversion equal to 0.5 are the best performing portfolios.

Table 7: Cumulative Return of Positional Momentum Portfolios Until October 2016

Risk Aversion	Estimated ρ's	Fitted ρ's
$\mathscr{A}_r = 0.5$	**134.18**	133.92
$\mathscr{A}_r = 1$	**67.137**	67.004
$\mathscr{A}_r = 3$	**22.437**	22.392
$\mathscr{A}_r = 5$	**13.497**	13.470
EW	0.087	

Note: Table 7 shows the cumulative return of optimal positional momentum portfolios with the inception date of April 2008 based on future ranks predicted with estimated $\hat{\rho}_{11,t},\ \hat{\rho}_{12,t}$ from VAR(1) model (equation 2.7) (Estimated ρ's) and the fitted value of estimated coefficients from equations (3.16) and (3.17) $\hat{\hat{\rho}}_{11,t+1},\ \hat{\hat{\rho}}_{12,t+1}$ (Fitted ρ's).

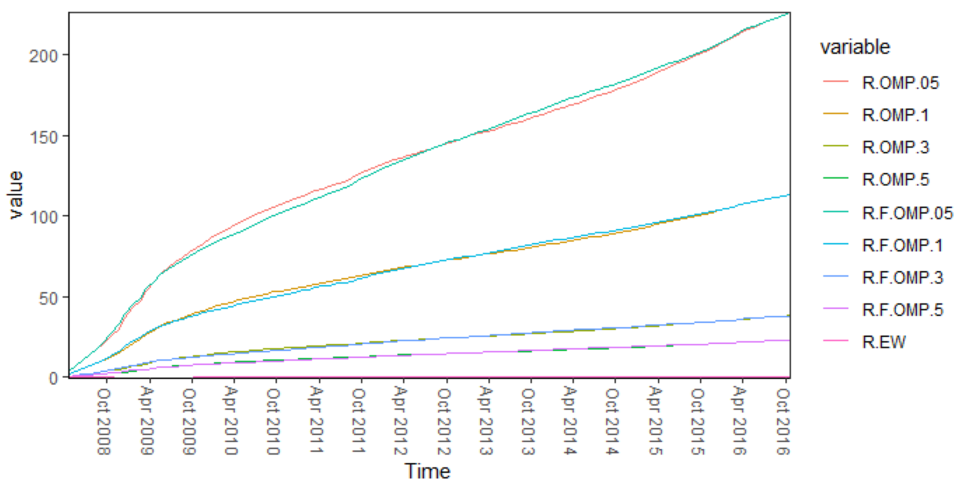

Figure 15: Time Series of Cumulative Returns of Positional Momentum Strategies

Figure 15 compares the time series of cumulative returns of positional momentum portfolios if one hold the portfolio until October 2016. The red, orange, olive and green line show the cumulative returns of optimal positional momentum portfolios computed from estimated parameters of VAR model when $\mathscr{A}_r = 0.5, 1, 3,$ and 5 respectively. The light green, light blue, blue and purple line show the cumulative returns of optimal positional momentum portfolios computed from fitted values of parameters from equations (3.16) and (3.17) when $\mathscr{A}_r = 0.5, 1, 3,$ and 5 respectively. The pink line shows the mean of the equally weighted portfolio.

For risk aversion of 0.5, the positional momentum portfolio based on fitted ρ's provides higher cumulative returns than the portfolio with estimated coefficients until May 2009, i.e. during the crisis and later, during the period January 2013 to February 2016 . The positional momentum portfolio based on estimated ρ's provides the highest return from

May 2009 until June 2012. The EW portfolio has the lowest cumulative returns.

5.2 Optimal Positional Liquid Portfolios

The optimal positional liquid portfolios contain stocks with allocations $\beta^*_{tv,t}$, defined as follows:

$$\beta^*_{tv,t} = \frac{1}{n}h + \frac{\rho_{21,t+1}(u_t - \overline{u}_t) + \rho_{22,t+1}(v_t - \overline{v}_t)}{\mathscr{A}_{tv}\sigma^2_{2,t+1}} \tag{5.65}$$

Table 8 shows the average of the time series of optimal positional liquid portfolios returns and their standard deviations and compares those returns with the equally weighted portfolio's return. Two types of positional liquid portfolios are considered again: type 1 portfolios rely on the future ranks predicted with estimated $\hat{\rho}_{21,t}, \hat{\rho}_{22,t}$ from the VAR(1) model (equation 2.7) and type 2 portfolios are computed with the fitted values $\hat{\hat{\rho}}_{21,t+1}, \hat{\hat{\rho}}_{22,t+1}$ obtained from factor model (equations (3.18) and (3.19)). The positional liquid portfolios are calculated for four different values of risk aversion ($\mathscr{A}_{tv} = 0.5, 1, 3, 5$).

Table 8: Summary of Positional Liquid Portfolios' Returns

	Estimated ρ's		Fitted ρ's	
Risk Aversion	**Mean**	**S-D**	**Mean**	**S-D**
$\mathscr{A}_{tv} = 0.5$	**3.795**	2.198	3.742	2.211
$\mathscr{A}_{tv} = 1$	**1.899**	1.100	1.873	1.106
$\mathscr{A}_{tv} = 3$	**0.636**	0.367	0.627	0.369
$\mathscr{A}_{tv} = 5$	**0.383**	0.221	0.378	0.222
	Mean		S-D	
EW	0.004		0.067	

Note: Table 8 shows the average of the time series return of the optimal positional liquid portfolios with the future ranks predicted with estimated $\hat{\rho}_{21,t}$, $\hat{\rho}_{22,t}$ from VAR(1) model (equation 2.7) (Estimated ρ's) and the with fitted value of estimated coefficients from equations (3.18) and (3.19) $\hat{\hat{\rho}}_{21,t+1}$, $\hat{\hat{\rho}}_{22,t+1}$ (Fitted ρ's).

The returns on both types of positional liquid portfolios with the estimated and fitted autoregressive coefficients are positive and higher than on the EW portfolio. Among them, portfolios based on the estimated ρ's from the VAR(1) model provide higher average returns than those based on fitted ρ's from equation (3.18) and (3.19), although their returns are very close. By comparing Tables 8 with 6, we find that the positional liquid portfolios provide higher average returns than the positional momentum portfolios. Hence, the positional portfolios of liquid assets give higher average returns than the positional portfolios of winners.

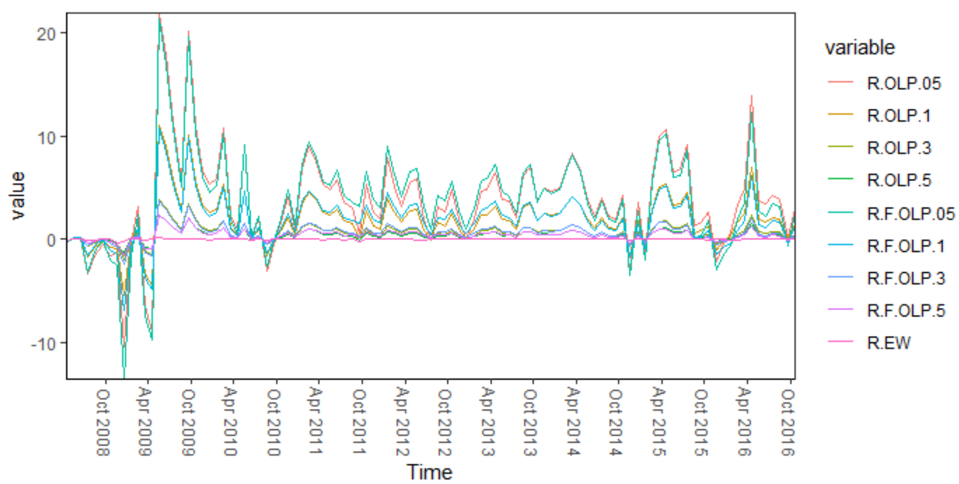

Figure 16: Time Series of Positional Liquid Strategies' Returns

Figure 16 compares the time series of positional liquid portfolios' returns. The red, orange, olive and green line show the returns of optimal positional liquid portfolios computed from estimated parameters of VAR model when $\mathscr{A}_r = 0.5, 1, 3,$ and 5 respectively. The light green, light blue, blue and purple line show the returns of optimal positional liquid portfolios computed from fitted values of parameters from equations (3.18) and (3.19) when $\mathscr{A}_r = 0.5, 1, 3,$ and 5 respectively. The pink line shows the mean of the equally weighted portfolio.

Figure 16 shows the time series of returns on positional liquid portfolios for different values of risk aversions. As expected, the positional liquid portfolios with risk aversion equal to 0.5 perform better than the other portfolios. The EW portfolio has the lowest returns in comparison to all portfolios considered. However, from November 2008 to February 2009, the EW portfolio provides higher returns than the positional liquid portfolios, which reached their lowest values during the crisis.

Table 9 shows the cumulative returns on positional liquid portfolios with the inception date of April 2008. We observe that, the positional liquid portfolios based on fitted ρ's outperform the other portfolios for all values of risk aversion considered. By comparing

Table 9 with Table 7, we find that a positional portfolio of liquid assets provided higher cumulative returns than a positional portfolios of winners.

Table 9: Cumulative Return of Positional Liquid Portfolios Until October 2016

Risk Aversion	Estimated ρ's	Fitted ρ's
$\mathscr{A}_{tv} = 0.5$	185.00	**187.07**
$\mathscr{A}_{tv} = 1$	92.544	**93.581**
$\mathscr{A}_{tv} = 3$	30.906	**31.251**
$\mathscr{A}_{tv} = 5$	18.578	**18.786**
EW	0.087	

Note: Table 9 shows the cumulative return of optimal positional liquid portfolios with the inception date of April 2008 based on future ranks predicted with estimated $\hat{\rho}_{21,t},\ \hat{\rho}_{22,t}$ from VAR(1) model (equation 2.7) (Estimated ρ's) and the fitted value of estimated coefficients from equations (3.18) and (3.19) $\hat{\hat{\rho}}_{21,t+1},\ \hat{\hat{\rho}}_{22,t+1}$ (Fitted ρ's).

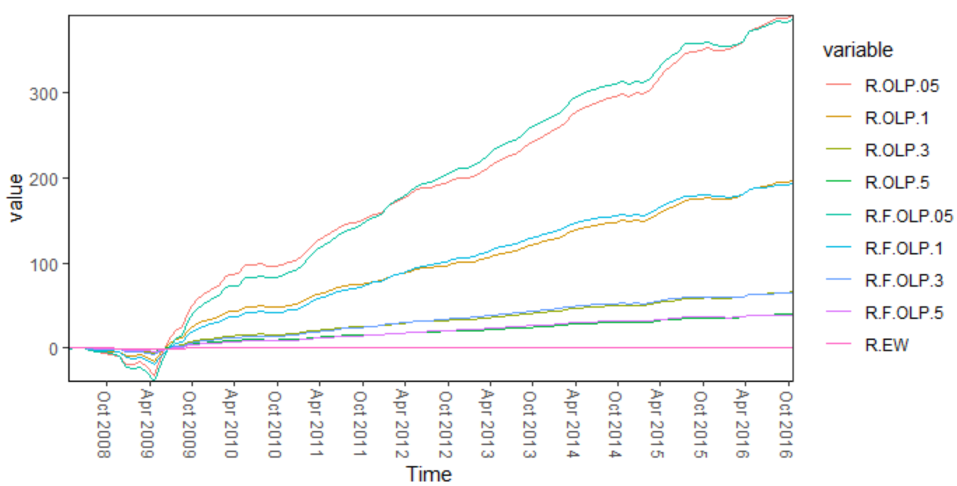

Figure 17: Time Series of Cumulative Returns of Positional Liquid Strategies

Figure 17 compares the time series of cumulative returns of positional liquid portfolios if one hold the portfolio until October 2016. The red, orange, olive and green line show the cumulative returns of optimal positional liquid portfolios computed from estimated parameters of VAR model when $\mathscr{A}_r = 0.5, 1, 3,$ and 5 respectively. The light green, light blue, blue and purple line show the cumulative returns of optimal positional liquid portfolios computed from fitted values of parameters from equations (3.18) and (3.19) when $\mathscr{A}_r = 0.5, 1, 3,$ and 5 respectively. The pink line shows the mean of the equally weighted portfolio.

Figure 17 shows the time series of cumulative returns on positional liquid portfolios from April 2008 to October 2016. Both positional liquid portfolios based on the estimated and fitted ρ's with risk aversion 0.5 outperform the other portfolios. At the beginning of the sampling period (April 2008 to July 2009), i.e. during the crisis, the cumulative returns on the positional liquid portfolios are below the returns on the EW portfolio. After July 2009, the cumulative returns on the positional liquid portfolios increase and remain higher than the cumulative returns on the EW portfolio.

5.3 Optimal Mixed Positional Portfolios

The optimal mixed positional portfolios contain assets with allocations β_t^* defined as follows:

$$\beta_t^* = \frac{1}{n}h + \frac{\mathscr{A}_r(\rho_{11,t+1}(u_t - \overline{u}_t) + \rho_{12,t+1}(v_t - \overline{v}_t))}{\Delta_t} + \frac{\mathscr{A}_{tv}(\rho_{21,t+1}(u_t - \overline{u}_t) + \rho_{22,t+1}(v_t - \overline{v}_t))}{\Delta_t} \tag{5.66}$$

where $\Delta_t = \mathscr{A}_r^2\sigma_{1,t+1}^2 + \mathscr{A}_{tv}^2\sigma_{2,t+1}^2 + 2\mathscr{A}_r\mathscr{A}_{tv}\sigma_{12,t+1}$. Table 7 compare the average returns and standard deviations on the positional mixed portfolios and on the EW portfolio. Again, two types of positional mixed portfolios are considered, one with the future ranks predicted with $\hat{\rho}_{jk,t}, j,k = 1,2$ estimated by rolling (equation) and another with $\hat{\hat{\rho}}_{jk,t+1}, j,k = 1,2$ predicted from the factor model (equation).

Table 10, shows the positional mixed portfolios computed for different values of risk aversion $\mathscr{A}_r = \mathscr{A}_{tv} = 0.5, 1, 3, 5$. Both types of positional mixed portfolios provide returns that are positive and higher than returns on the EW portfolio. Like in the case of optimal positional momentum portfolios, the positional mixed portfolios based on estimated ρ's have higher returns than those based on fitted ρ's.

By comparing Tables 10,8 and 6, we find that the positional liquid portfolios provide higher average returns than the positional mixed portfolios. However, the average returns on the positional mixed portfolios are higher than on the positional momentum portfolios. Hence, the positional portfolio which contains liquid winners provides a higher average return than a positional portfolio of winners. In fact, by considering the liquidity along with the returns, we can improve the performance of positional portfolios.

Table 10: Summary of Positional Mixed Portfolios' returns

	Estimated ρ's		Fitted ρ's	
Risk Aversion	**Mean**	**S-D**	**Mean**	**S-D**
$\mathscr{A}_r = \mathscr{A}_{tv} = 0.5$	**2.983**	2.323	2.954	2.295
$\mathscr{A}_r = \mathscr{A}_{tv} = 1$	**1.493**	1.177	1.479	1.164
$\mathscr{A}_r = \mathscr{A}_{tv} = 3$	**0.500**	0.416	0.496	0.412
$\mathscr{A}_r = \mathscr{A}_{tv} = 5$	**0.302**	0.266	0.299	0.263
	Mean		S-D	
EW	0.004		0.067	

Note: Table 10 shows the average of the time series return of the optimal positional mixed portfolios with the future ranks predicted with estimated $\hat{\rho}_{11,t}$, $\hat{\rho}_{21,t}$, $\hat{\rho}_{21,t}$ and $\hat{\rho}_{22,t}$ from VAR(1) model (equation 2.7) (Estimated ρ's) and the with fitted value of estimated coefficients from equations (3.16) to (3.19) $\hat{\hat{\rho}}_{11,t+1}$, $\hat{\hat{\rho}}_{12,t+1}$, $\hat{\hat{\rho}}_{21,t+1}$ and $\hat{\hat{\rho}}_{22,t+1}$ (Fitted ρ's).

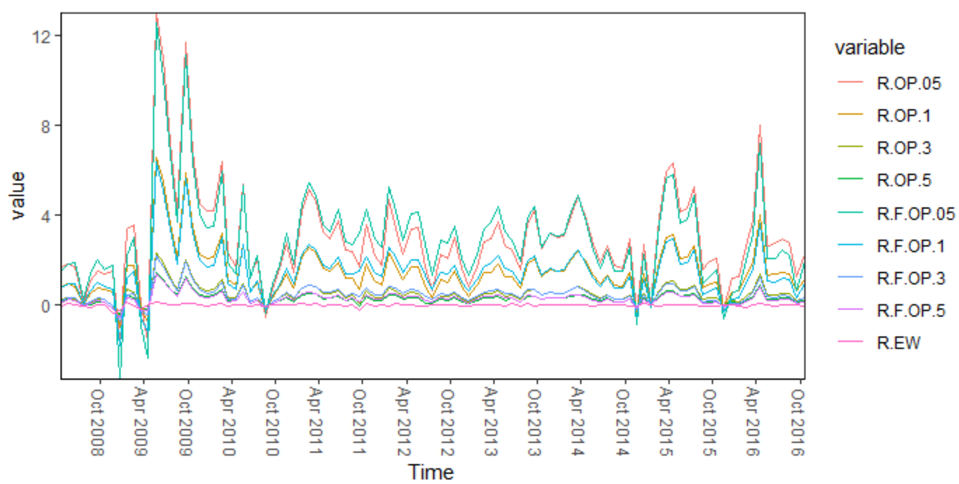

Figure 18: Time Series of Mixed Positional Strategies' Returns

Figure 18 compares the time series of the positional mixed portfolios' returns. The red, orange, olive and green line show the returns of optimal positional mixed portfolios computed from estimated parameters of VAR model when $\mathscr{A}_r = \mathscr{A}_{tv} = 0.5,\ 1\ ,3,$ and 5 respectively. The light green, light blue, blue and purple line show the returns of optimal positional mixed portfolios computed from fitted values of parameters from equations (3.16) to (3.19) when $\mathscr{A}_r = \mathscr{A}_{tv} = 0.5,\ 1\ ,3,$ and 5 respectively. The pink line shows the mean of the equally weighted portfolio.

Moreover, a positional portfolios of liquid assets provide even higher average return than a positional portfolios of liquid winners. Figure 18 shows the time series of returns on the positional mixed portfolios. We observe very similar patterns as in Figure 16. The portfolios based on estimated ρ's and fitted ρ's with risk aversion of 0.5 provide the highest returns, while the EW portfolio provides the lowest returns. During the crisis we see that all portfolios' returns drop lower than EW portfolio and provide negative returns. Also in August 2010, December 2014 and 2015 we observe the negative returns for all these portfolios. While in July and August 2009, between December 2014 to September 2015 and in March 2016, we observe high and positive returns for positional portfolios with $\mathscr{A}_r = \mathscr{A}_t v = 0.5,\ 1$.

Table 11 shows the cumulative returns on positional mixed portfolios with the inception date of April 2008 until October 2016. We observe that cumulative return on positional mixed portfolios based on fitted ρ's are higher than those based on estimated ρ's. Again, higher risk aversion value provides lower cumulative return. In both Table 11 and 9 the positional portfolios based on fitted ρ's provide higher return than those based on estimated ρ's. While in Table 7, the positional portfolios based on estimated ρ's yield in higher returns. By comparing Table 11,9 and 7, we observe that a positional portfolio based on liquid assets outperforms other positional portfolios. However, a positional portfolios based on liquid winners has higher cumulative return than the positional portfolio based on just winner stocks.

Figure 19 shows the cumulative returns on the positional mixed portfolios from April 2008 until October 2016. For risk aversion of 0.5, the positional portfolio of liquid winners obtained from fitted ρ's provides the highest return until January 2009 (crisis) and between March 2012 and April 2016. From January 2009 to March 2012 and from May 2016 until the end of the sampling period, the positional mixed portfolio based on estimated ρ's has the higher returns. Between March 2012 and May 2016 the positional mixed portfolios based on fitted ρ's with $\mathscr{A} = 0.5$ outperforms other portfolios' returns. The second best returns belong to the positional mixed portfolios when $\mathscr{A} = 1$. Again, between January 2009 to March 2012 and from May 2016 until the end of the sampling period, the positional mixed portfolio based on estimated ρ's has the higher returns, while between March 2012 and May 2016 the positional mixed portfolios based on fitted ρ's with $\mathscr{A} = 0.5$ outperforms other portfolios' returns.

Table 11: Cumulative Return of Positional Mixed Portfolios Until October 2016

Risk Aversion	Estimated ρ's	Fitted ρ's
$\mathscr{A}_r = \mathscr{A}_{tv} = 0.5$	159.13	**160.02**
$\mathscr{A}_r = \mathscr{A}_{tv} = 1$	79.610	**80.05**
$\mathscr{A}_r = \mathscr{A}_{tv} = 3$	26.595	**26.743**
$\mathscr{A}_r = \mathscr{A}_{tv} = 5$	15.991	**16.081**
EW	0.087	

Note: Table 11 shows the cumulative return of optimal positional mixed portfolios with the inception date of April 2008 based on future ranks predicted with estimated $\hat{\rho}_{11,t}$, $\hat{\rho}_{12,t}$, $\hat{\rho}_{21,t}$ and $\hat{\rho}_{22,t}$ from VAR(1) model (equation 2.7) (Estimated ρ's) and the fitted value of estimated coefficients from equations (3.16) to (3.19) $\hat{\hat{\rho}}_{11,t+1}$, $\hat{\hat{\rho}}_{12,t+1}$, $\hat{\hat{\rho}}_{21,t+1}$ and $\hat{\hat{\rho}}_{22,t+1}$ (Fitted ρ's).

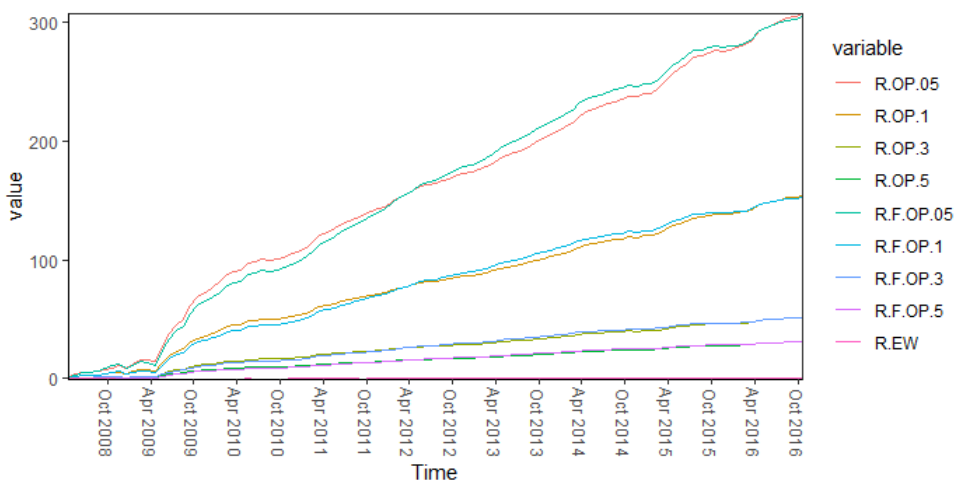

Figure 19: Time Series of Cumulative Returns of Positional Mixed Strategies

Figure 19 compares the time series of cumulative returns of positional mixed portfolios if one hold the portfolio until October 2016. The red, orange, olive and green line show the cumulative returns of optimal positional mixed portfolios computed from estimated parameters of VAR model when $\mathscr{A}_r = \mathscr{A}_{tv} = 0.5,\ 1\ ,3,$ and 5 respectively. The light green, light blue, blue and purple line show the cumulative returns of optimal positional mixed portfolios computed from fitted values of parameters from equations (3.16) to (3.19) when $\mathscr{A}_r = \mathscr{A}_{tv} = 0.5,\ 1\ ,3,$ and 5 respectively. The pink line shows the mean of the equally weighted portfolio.

Let us now assume that $\mathscr{A}_r \neq \mathscr{A}_{tv}$. Below, we examine the positional mixed portfolios with different values of risk aversion. First, we consider $\mathscr{A}_r$ fixed and compute the positional mixed portfolios for different values of $\mathscr{A}_{tv}$. Next, we consider $\mathscr{A}_{tv}$ fixed and compute the positional mixed portfolios for different values of $\mathscr{A}_r$.

Figure 20 shows the time series of returns on the positional mixed portfolios when $\mathscr{A}_r = 0.5$. The positional mixed portfolios with risk aversion equal to 0.5 outperforming the other portfolios. The EW portfolio has the lowest returns. Same as Figure 18, during the crisis, August 2010, December 2014, March and December 2015 we observe that all the portfolios returns have been dropped to the negative value. While their returns reached to their positive peaks on June and October 2009 and April 2016.

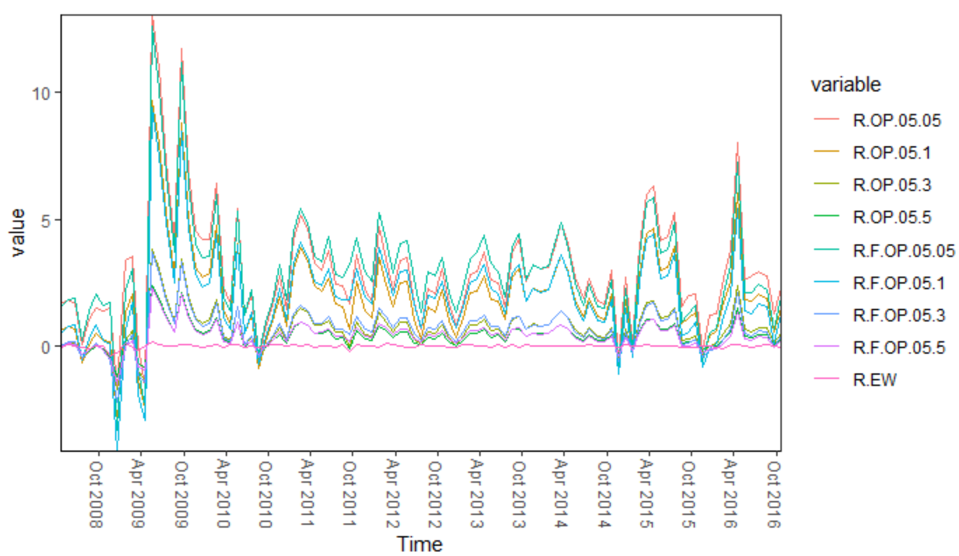

Figure 20: Time Series of Mixed Positional Strategies' Returns, $\mathscr{A}_r = 0.5$

Figure 20 compares the time series of positional mixed portfolios' returns for $\mathscr{A}_r = 0.5$. The red, orange, olive and green line show the returns of optimal positional mixed portfolios computed from estimated parameters of VAR model when $\mathscr{A}_{tv} = 0.5,\ 1,\ 3$ and 5 respectively. The light green, light blue, blue and purple line show the returns of optimal positional mixed portfolios computed from fitted values of parameters from equations (3.16) to (3.19) when $\mathscr{A}_{tv} = 0.5,\ 1,\ 3$ and 5 respectively. The pink line shows the mean of the equally weighted portfolio.

Figure 21 shows the time series of returns on the positional mixed portfolios for $\mathscr{A}_r = 1$. From April 2008 to January 2010 the positional mixed portfolios of both types with $\mathscr{A}_{tv} = 0.5$ outperforms the other portfolios. Between February 2010 to June 2014 the positional mixed portfolios based on fitted ρ's with $\mathscr{A}_{tv} = 0.5,\ 1$ provide the highest returns. After June 2014, the positional mixed portfolios based on estimated ρ's with $\mathscr{A}_{tv} = 0.5,\ 1$ have the highest returns. The patterns of these two figures are very close and we observe that by reducing the value $\mathscr{A}_{tv}$ the portfolios' returns decreased.

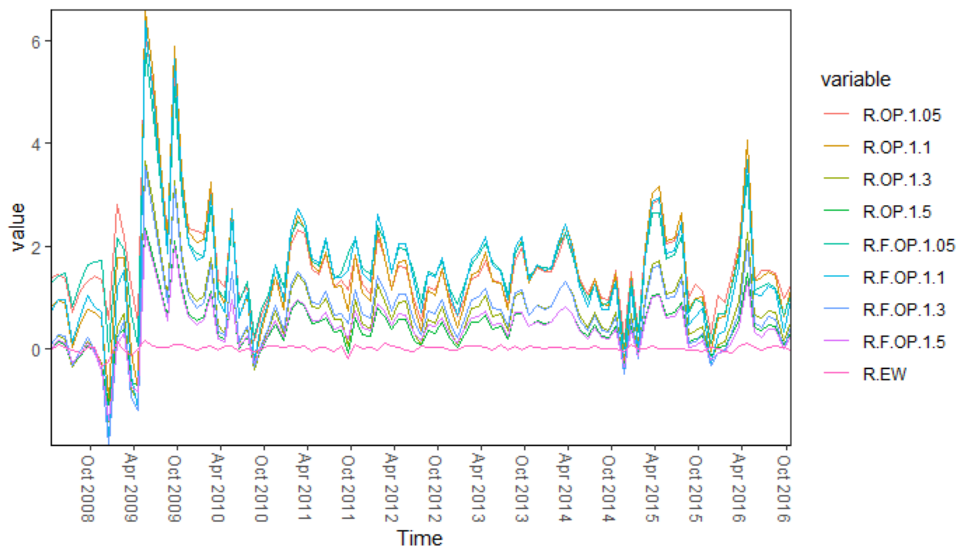

Figure 21: Time Series of Mixed Positional Strategies' Returns, $\mathscr{A}_r = 1$

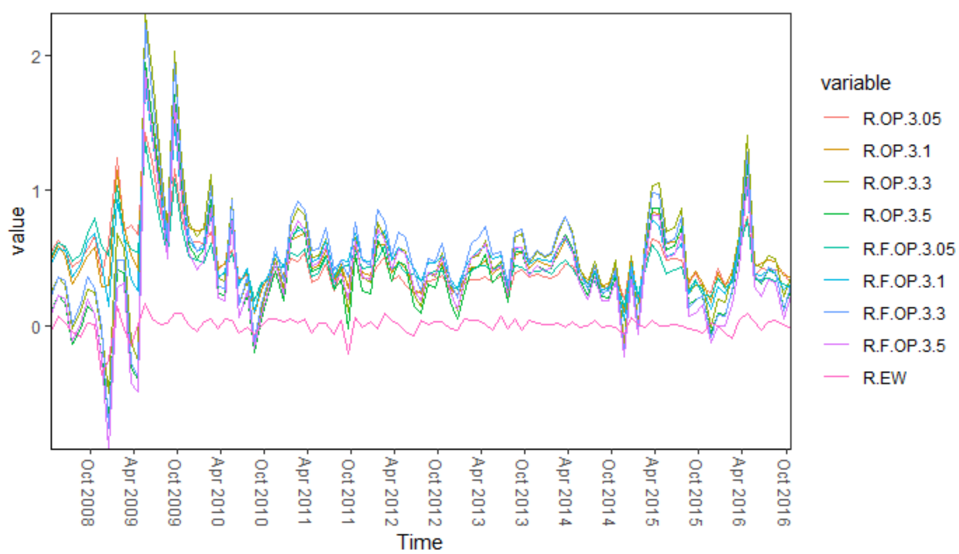

Figure 22: Time Series of Mixed Positional Strategies' Returns, $\mathscr{A}_r = 3$

Figure 21 and 22 compare the time series of positional mixed portfolios' returns for $\mathscr{A}_r = 1$ and 3 respectively. The red, orange, olive and green line show the returns of optimal positional mixed portfolios computed from estimated parameters of VAR model when $\mathscr{A}_{tv} = 0.5,\ 1,\ 3$ and 5 respectively. The light green, light blue, blue and purple line show the returns of optimal positional mixed portfolios computed from fitted values of parameters from equations (3.16) to (3.19) when $\mathscr{A}_{tv} = 0.5,\ 1,\ 3$ and 5 respectively. The pink line shows the mean of the equally weighted portfolio.

Figure 22 shows the time series of the returns on the positional mixed portfolios for $\mathscr{A}_r = 3$. The variations of the returns on Figure 22 is more than what we observe in Figure 21. The returns on positional mixed portfolios based on estimated and fitted ρ's are very close. Most of the time the positional mixed portfolios obtained from fitted ρ's provide higher returns especially those with $\mathscr{A}_{tv} = 1,\ 3$.

The peaks and falls in their returns are same as what we observe in Figure 21. During the crisis, August 2010, December 2014, March and December 2015 all returns falls to the negative values while on June and October 2009 and April 2016 they reach to their highest values.

Figure 23 shows the time series of the positional mixed portfolios' return for $\mathscr{A}_r = 5$. In contrast to what we observed in the previous Figures, the positional mixed portfolios obtained from both estimated and fitted 's with $\mathscr{A}_t v = 5$ provide this time returns higher than the other portfolios. This result is illustrated further by the average and cumulative returns given in Table 12 below.

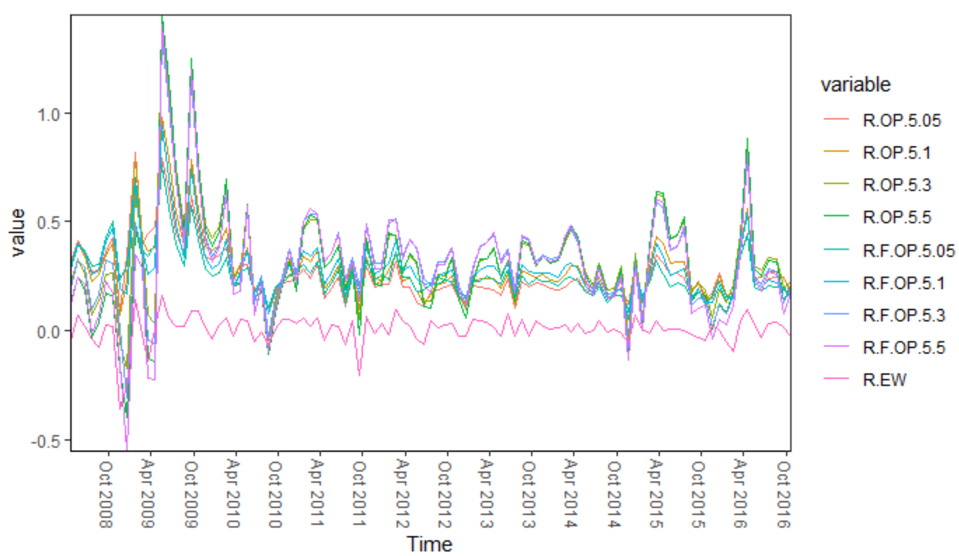

Figure 23: Time Series of Mixed Positional Strategies' Returns, $\mathscr{A}_r = 5$

Figure 23 compares the time series of positional mixed portfolios' returns for $\mathscr{A}_r = 5$. The red, orange, olive and green lines show the returns of optimal positional mixed portfolios computed from estimated parameters of VAR model when $\mathscr{A}_{tv} = 0.5,\ 1,\ 3$ and 5 respectively. The light green, light blue, blue and purple lines show the returns of optimal positional mixed portfolios computed from fitted values of parameters from equations (3.16) to (3.19) when $\mathscr{A}_{tv} = 0.5,\ 1,\ 3$ and 5 respectively. The pink line shows the mean of the equally weighted portfolio.

Table 12, reports the average and cumulative returns and standard deviations for positional mixed portfolios in 4 panels, which present the results obtained when $\mathscr{A}_r$ is constant and $\mathscr{A}_{tv}$ is allowed to vary. The panels show the results for $\mathscr{A}_r = 0.5, 1, 3, 5$, arranged from lowest to highest risk aversion. For all values of risk aversion $\mathscr{A}_r = 0.5, 1, 3, 5$ displayed in the four panels, the mixed portfolios based on estimated ρ's provide higher average returns, while the positional mixed portfolios based on fitted ρ's provide higher cumulative returns.

The cumulative returns of these two strategies are very close. In the bottom panel we observe that the positional mixed portfolios obtained from fitted ρ's have higher cumulative returns except for $\mathscr{A}_{tv} = 0.5$, where the portfolio obtained from estimated ρ's provides a higher return. Moreover, for $\mathscr{A}_r = 0.5$ (top panel) and $\mathscr{A}_r = 1$ the highest average and cumulative returns are on portfolios with $\mathscr{A}_{tv} = 0.5$. In contrast, for $\mathscr{A}_r = 3$, we observe that the positional mixed portfolios of both types (estimated and fitted ρ's) higher with $\mathscr{A}_{tv}$ equal to 1 and 3 provide higher average and cumulative returns than with $\mathscr{A}_{tv} = 0.5$. The risk-return trade-off is reversed further for $\mathscr{A}_r = 5$ displayed in the bottom panel. Among these portfolios, those with $\mathscr{A}_{tv} = 0.5$ have the lowest average returns and the portfolios with $\mathscr{A}_{tv} = 5$ produce the highest average and cumulative returns.

By comparing the four panels, we observe that when the value of risk aversion $\mathscr{A}_r$ increases, the average and cumulative returns decrease. In terms of average returns, the positional mixed portfolios obtained from estimated ρ's outperform those obtained from fitted ρ's. However, the positional mixed portfolios obtained from fitted ρ's provide higher cumulative returns for all values of $\mathscr{A}_r$. The only exception is $\mathscr{A}_r = 5$ and $\mathscr{A}_{tv} = 0.5$ where the portfolio obtained from estimated ρ's provides a higher cumulative return than the one obtained from fitted ρ's.

Table 12: Mixed Positional Portfolios' Returns, $\mathscr{A}_r = const$, $\mathscr{A}_{tv}$ vary

$\mathscr{A}_r = 0.5$	**Average Return**		**Cumulative Return**	
Risk Aversion	**Estimated ρ's**	**Fitted ρ's**	**Estimated ρ's**	**Fitted ρ's**
$\mathscr{A}_{tv} = 0.5$	**2.983**	2.954	159.13	**160.02**
$\mathscr{A}_{tv} = 1$	**1.960**	1.938	101.05	**101.82**
$\mathscr{A}_{tv} = 3$	**0.679**	0.671	33.780	**34.109**
$\mathscr{A}_{tv} = 5$	**0.401**	0.396	19.758	**19.960**
$\mathscr{A}_r = 1$	**Average Return**		**Cumulative Return**	
Risk Aversion	**Estimated ρ's**	**Fitted ρ's**	**Estimated ρ's**	**Fitted ρ's**
$\mathscr{A}_{tv} = 0.5$	**1.626**	2.954	90.094	**90.395**
$\mathscr{A}_{tv} = 1$	**1.960**	1.479	79.610	**80.057**
$\mathscr{A}_{tv} = 3$	**0.683**	0.675	34.625	**34.922**
$\mathscr{A}_{tv} = 5$	**0.411**	0.406	20.500	**20.694**
$\mathscr{A}_r = 3$	**Average Return**		**Cumulative Return**	
Risk Aversion	**Estimated ρ's**	**Fitted ρ's**	**Estimated ρ's**	**Fitted ρ's**
$\mathscr{A}_{tv} = 0.5$	**0.459**	0.457	26.694	**26.706**
$\mathscr{A}_{tv} = 1$	**0.518**	0.515	29.245	**29.306**
$\mathscr{A}_{tv} = 3$	**0.500**	0.496	26.595	**26.743**
$\mathscr{A}_{tv} = 5$	**0.379**	0.375	19.617	**19.757**
$\mathscr{A}_r = 5$	**Average Return**		**Cumulative Return**	
Risk Aversion	**Estimated ρ's**	**Fitted ρ's**	**Estimated ρ's**	**Fitted ρ's**
$\mathscr{A}_{tv} = 0.5$	**0.257**	0.256	**15.142**	15.135
$\mathscr{A}_{tv} = 1$	**0.286**	0.284	16.444	**16.457**
$\mathscr{A}_{tv} = 3$	**0.330**	0.327	17.993	**18.064**
$\mathscr{A}_{tv} = 5$	**0.302**	0.299	15.991	**16.081**
	Mean		S-D	
EW	0.0043		0.087	

Note: Table 12 shows the average and cumulative return of optimal positional mixed portfolios with the inception date of April 2008 based on future ranks predicted with estimated $\hat{\rho}_{11,t}$, $\hat{\rho}_{12,t}$, $\hat{\rho}_{21,t}$ and $\hat{\rho}_{22,t}$ from VAR(1) model (equation 2.7) (Estimated ρ's) and the fitted value of estimated coefficients from equations (3.16) to (3.19) $\hat{\hat{\rho}}_{11,t+1}$, $\hat{\hat{\rho}}_{12,t+1}$, $\hat{\hat{\rho}}_{21,t+1}$ and $\hat{\hat{\rho}}_{22,t+1}$ (Fitted ρ's), considering $\mathscr{A}_r$ constant.

Figure 24 shows the time series of positional mixed portfolios with $\mathscr{A}_{tv} = 0.5$. This figure is very similar to Figure 20. The highest returns belong to the positional mixed portfolios of both types with $\mathscr{A}_r = 0.5$ and the lowest returns belongs to EW. We also observe that by increasing the value of $\mathscr{A}_r$ the returns of these portfolios decreased. The returns on these portfolios reached to their peaks on August 2010, December 2014, March and December 2015. While they dropped to their lowest values during crisis.

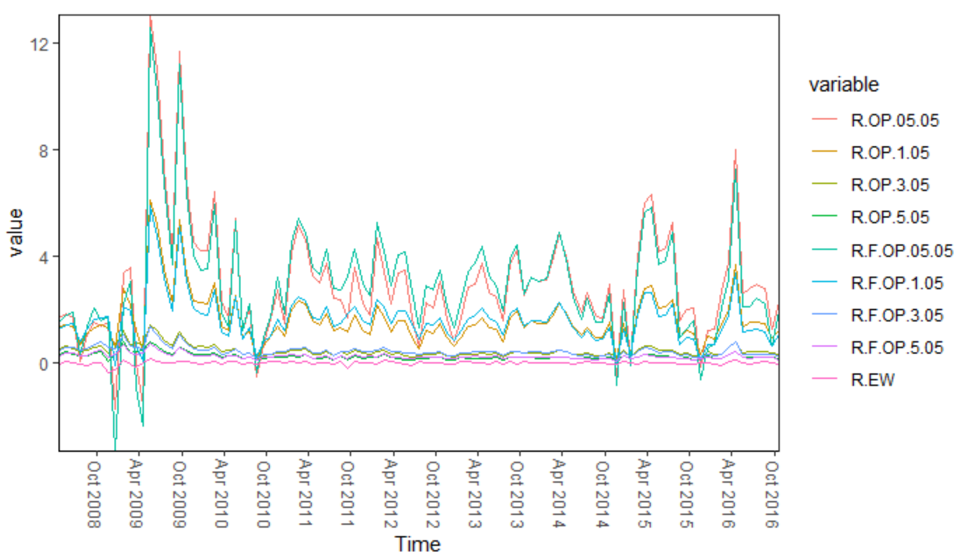

Figure 24: Time Series of Mixed Positional Strategies' Returns, $\mathscr{A}_{tv} = 0.5$

Figure 24 compares the time series of positional mixed portfolios' returns for $\mathscr{A}_{tv} = 0.5$. The red, orange, olive and green lines show the returns of optimal positional mixed portfolios computed from estimated parameters of VAR model when $\mathscr{A}_r = 0.5,\ 1,\ 3$ and 5 respectively. The light green, light blue, blue and purple lines show the returns of optimal positional mixed portfolios computed from fitted values of parameters from equations (3.16) to (3.19) when $\mathscr{A}_r = 0.5,\ 1,\ 3$ and 5 respectively. The pink line shows the mean of the equally weighted portfolio.

Figure 25 shows the time series of the positional mixed portfolios for $\mathscr{A}_{tv} = 1$. Similar to Figure 24, the portfolios with the lowest risk aversion $\mathscr{A}_r = 0.5$ provide the highest returns and the EW portfolio provides the lowest return. Figure 26 shows the time series of positional mixed portfolios when $\mathscr{A}_{tv} = 3$. These time series of returns display the same patterns as those in Figure 25. We notice the risk-return trade-off as well, as higher risks yield higher returns.

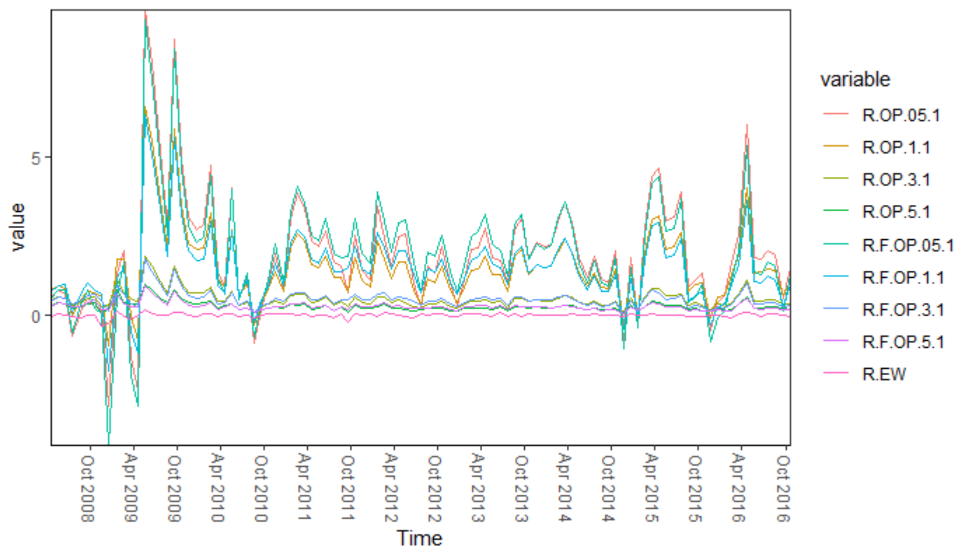

Figure 25: Time Series of Mixed Positional Strategies' Returns, $\mathscr{A}_{tv} = 1$

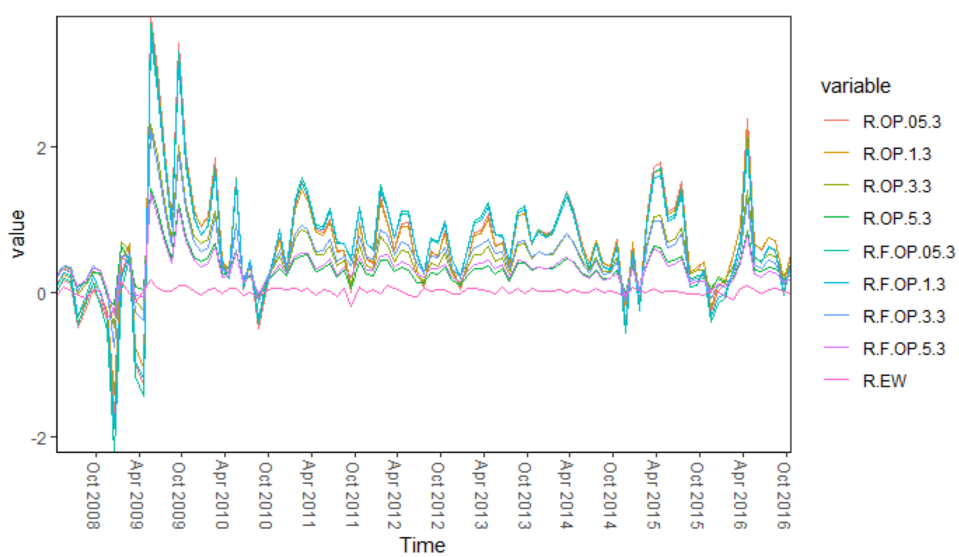

Figure 26: Time Series of Mixed Positional Strategies' Returns, $\mathscr{A}_{tv} = 3$

Figure 25 and 26 compare the time series of positional mixed portfolios' returns for $\mathscr{A}_{tv} = 1$ and 3 respectively. The red, orange, olive and green lines show the returns of optimal positional mixed portfolios computed from estimated parameters of VAR model when $\mathscr{A}_r = 0.5,\ 1,\ 3$ and 5 respectively. The light green, light blue, blue and purple lines show the returns of optimal positional mixed portfolios computed from fitted values of parameters from equations (3.16) to (3.19) when $\mathscr{A}_r = 0.5,\ 1,\ 3$ and 5 respectively. The pink line shows the mean of the equally weighted portfolio.

Figure 27 provides the time series of the positional mixed portfolios when $\mathscr{A}_{tv} = 5$. Similarly to the previous figures, we observe the risk-return trade-off. The higher risk aversion, the lower the returns. The returns on all the positional portfolios are very close, while the EW portfolio still provides the lowest return. Again during crisis all portfolios provide negative returns, while on August and December 2009 they reached to their highest values respectively.

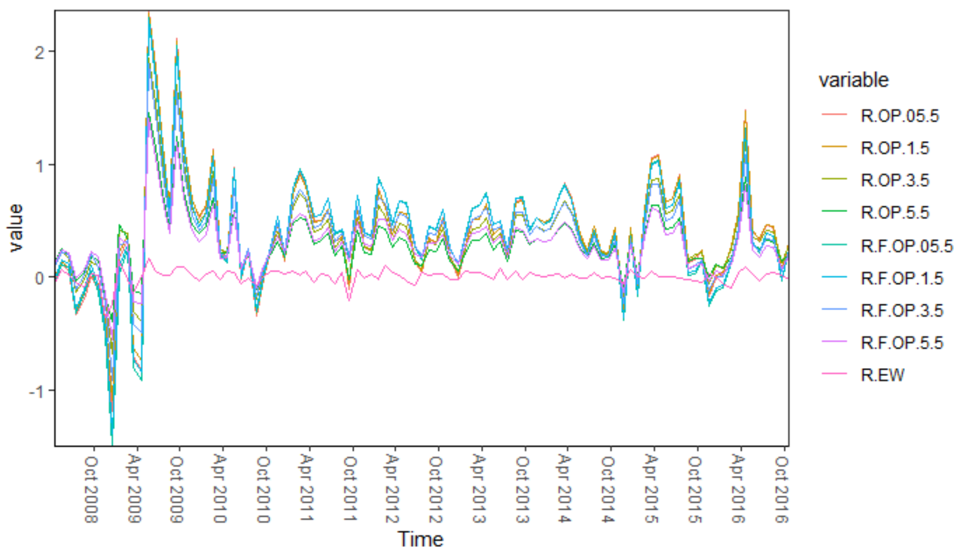

Figure 27: Time Series of Mixed Positional Strategies' Returns, $\mathscr{A}_{tv} = 5$

Figure 27 compares the time series of positional mixed portfolios' returns for $\mathscr{A}_{tv} = 5$. The red, orange, olive and green lines show the returns of optimal positional mixed portfolios computed from estimated parameters of VAR model when $\mathscr{A}_r = 0.5,\ 1,\ 3$ and 5 respectively. The light green, light blue, blue and purple lines show the returns of optimal positional mixed portfolios computed from fitted values of parameters from equations (3.16) to (3.19) when $\mathscr{A}_r = 0.5,\ 1,\ 3$ and 5 respectively. The pink line shows the mean of the equally weighted portfolio.

Table 13, shows the average and cumulative returns on positional mixed portfolios for fixed values of $\mathscr{A}_{tv}$ and varying $\mathscr{A}_r$. The results on portfolios with risk aversion $\mathscr{A}_{tv} = 0.5$ are displayed in the top panel, followed by the results for $\mathscr{A}_{tv} = 1, 3$ and 5 are displayed in four panels. Each panel presents the returns on portfolios with $\mathscr{A}_r = 0.5, 1, 3, 5$, for a given fixed value of $\mathscr{A}_{tv}$.

Table 13: Summary of Mixed Positional Portfolios' Returns, $\mathscr{A}_{tv} = 0.5$

$\mathscr{A}_{tv} = 0.5$	Average Return		Cumulative Return	
Risk Aversion	**Estimated ρ's**	**Fitted ρ's**	**Estimated ρ's**	**Fitted ρ's**
$\mathscr{A}_r = 0.5$	**2.983**	2.954	159.134	**160.027**
$\mathscr{A}_r = 1$	**1.626**	1.613	90.094	**90.395**
$\mathscr{A}_r = 3$	**0.459**	0.457	26.694	**26.706**
$\mathscr{A}_r = 5$	**0.257**	0.256	**15.142**	15.135
$\mathscr{A}_{tv} = 1$	Average Return		Cumulative Return	
Risk Aversion	**Estimated ρ's**	**Fitted ρ's**	**Estimated ρ's**	**Fitted ρ's**
$\mathscr{A}_r = 0.5$	**1.960**	1.938	101.054	**101.827**
$\mathscr{A}_r = 1$	**1.493**	1.479	79.610	**80.057**
$\mathscr{A}_r = 3$	**0.518**	0.515	29.245	**29.306**
$\mathscr{A}_r = 5$	**0.286**	0.284	16.444	**16.457**
$\mathscr{A}_{tv} = 3$	Average Return		Cumulative Return	
Risk Aversion	**Estimated ρ's**	**Fitted ρ's**	**Estimated ρ's**	**Fitted ρ's**
$\mathscr{A}_r = 0.5$	**0.679**	0.671	33.780	**34.109**
$\mathscr{A}_r = 1$	**0.683**	0.675	34.625	**34.922**
$\mathscr{A}_r = 3$	**0.500**	0.496	26.595	**26.743**
$\mathscr{A}_r = 5$	**0.330**	0.327	17.993	**18.0641**
$\mathscr{A}_{tv} = 5$	Average Return		Cumulative Return	
Risk Aversion	**Estimated ρ's**	**Fitted ρ's**	**Estimated ρ's**	**Fitted ρ's**
$\mathscr{A}_r = 0.5$	**0.401**	0.396	19.758	**19.960**
$\mathscr{A}_r = 1$	**0.411**	0.406	20.500	**20.694**
$\mathscr{A}_r = 3$	**0.379**	0.375	19.617	**19.757**
$\mathscr{A}_r = 5$	**0.302**	0.299	15.991	**16.081**
	Mean		S-D	
EW	0.0043		0.087	

Note: Table 13 shows the average and cumulative return of optimal positional mixed portfolios with the inception date of April 2008 based on future ranks predicted with estimated $\hat{\rho}_{11,t}$, $\hat{\rho}_{12,t}$, $\hat{\rho}_{21,t}$ and $\hat{\rho}_{22,t}$ from VAR(1) model (equation 2.7) (Estimated ρ's) and the fitted value of estimated coefficients from equations (3.16) to (3.19) $\hat{\hat{\rho}}_{11,t+1}$, $\hat{\hat{\rho}}_{12,t+1}$, $\hat{\hat{\rho}}_{21,t+1}$ and $\hat{\hat{\rho}}_{22,t+1}$ (Fitted ρ's), considering $\mathscr{A}_{tv}$ constant.

In all panels, the positional mixed portfolios obtained from estimated ρ's provide higher average returns. In terms of cumulative returns, the portfolios obtained from fitted ρ's outperform the others portfolios. There is one exception, however. In the top panel, we find that for $\mathscr{A}_{tv} = 0.5$ and $\mathscr{A}_r = 5$, the portfolios with estimated coefficients provide higher cumulative returns. Moreover, higher risks yield higher average and cumulative returns, as observed earlier.

By comparing the results in all Tables provided in Section 5, we find that the positional liquid portfolios provide the highest average and cumulative returns, as compared to the other strategies. Hence, a positional portfolio of liquid assets provides higher returns than a positional portfolio of winners. In addition, we find that the positional mixed portfolios provide higher average and cumulative returns than the positional momentum portfolios. In other words, a positional portfolio of liquid winners provides higher average and cumulative returns than a positional portfolio containing just the winner stocks.

6 Conclusion

This paper introduced new positional investment strategies that maximize investors positional utility from portfolios of assets with expected high return ranks, high liquidity ranks and high combined return-liquidity ranks. The optimal allocation vectors are computed from return and volume change ranks modelled as a panel VAR with time varying coefficients. We show that the autoregressive VAR parameters can be well approximated by linear functions of auto- and cross- correlations of the returns and volume change series of the SPDR tracking portfolio.

The empirical results indicate that all positional portfolios provide positive average and cumulative return. The positional liquid portfolios outperform the positional mixed and momentum portfolios respectively. Also, we observe that for higher risk aversion values, the average and cumulative returns on the positional portfolios decrease. In terms of average returns, the positional portfolios obtained from estimated coefficients $\hat{\rho}_{11,t}$, $\hat{\rho}_{12,t}$, $\hat{\rho}_{21,t}$ and $\hat{\rho}_{22,t}$ from VAR(1) model (equation 2.7) outperform the other portfolios. In terms of cumulative returns, the positional portfolios obtained from fitted values of coefficients based on auto- and cross- correlation of SPDR ($\hat{\hat{\rho}}_{11,t+1}$, $\hat{\hat{\rho}}_{12,t+1}$, $\hat{\hat{\rho}}_{21,t+1}$ and $\hat{\hat{\rho}}_{22,t+1}$) provide higher returns.

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Appendices

A Histograms of Auto and Cross-Correlations of SPDR Return and Trade volume Changes and of the Series of Estimated Autoregressive Coefficients

The following figures show the histograms of the time series of sample auto- cross- correlations of SPDR return and trade volume changes and of the time series of autoregressive coefficients $\hat{\rho}_{jk,t},\ j = 1, 2,\ k = 1, 2,\ t = 1, ..., T$ of the VAR(1) model (equation 2.10). All series are estimated by rolling with a window of 9 years over the sampling period.

Figure B.1, shows that sample auto-correlations of r_t^S take values mostly between -0.1 and 0.05 and their density is asymmetric with a long left tail. The series $\hat{\rho}_{11t}$ takes smaller values between -0.01 and 0.006, and has a symmetric density. Figure B.2 shows that the cross-correlations of r_t^S,tv_{t-1}^S take values mostly between -0.1 and 0.2 and their density displays asymmetry in the right tail. The series $\hat{\rho}_{21t}$ takes only positive values, with the most frequently observed values in the interval (0.001,0.002). The density of cross-correlations of tv_t^S,r_{t-1}^S given in Figure B,3 is almost bimodal. These cross-correlations take positive values only. The density of $\hat{\rho}_{21t}$ is similar in shape but its support includes small positive and negative values. Figure B.4 shows the density of sample auto-correlations of tv_t^S, which take negative values. Their density is symmetric and bell-shaped. The density of $r\hat{h}o_{22t}$, which also take negative values only, is asymmetric with a long left tail.

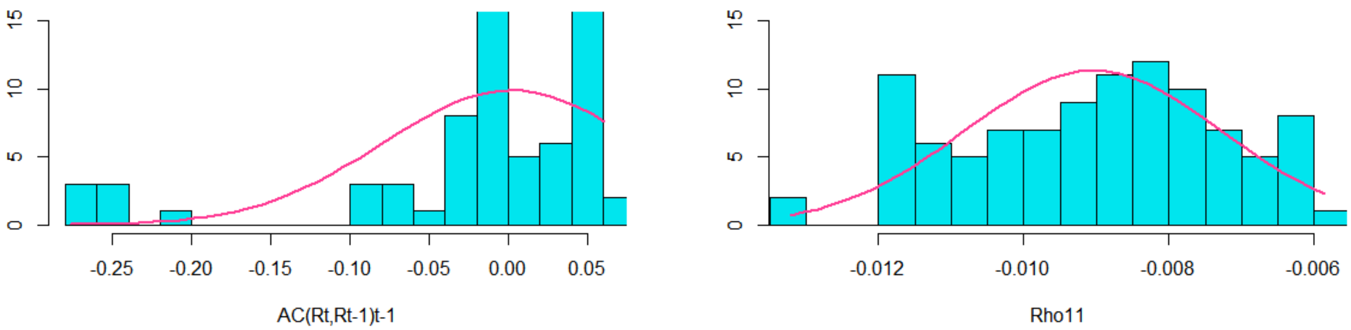

Figure B.1: Histograms of autocorrelations at lag one of r_t^S and of $\hat{\rho}_{11t}$

Figure B.1 compares the histograms of autocorrelations at lag one of SPDR's returns and the estimated $\hat{\rho}_{11t}$ from equation (2.10). In both plots the red line shows the kernel density estimates.

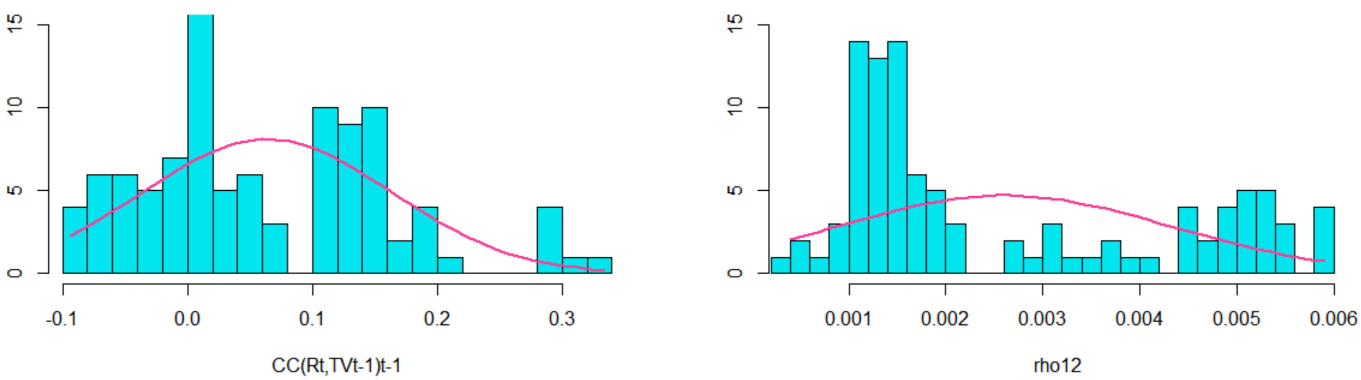

Figure B.2: Histograms of cross-correlation of r_t^S, tv_{t-1}^S and of $\hat{\rho}_{12t}$

Figure B.2 compares the histograms of cross-correlations of SPDR's r_t^S, tv_{t-1}^S and the estimated $\hat{\rho}_{12t}$ from equation (2.10). In both plots the red line shows the kernel density estimates.

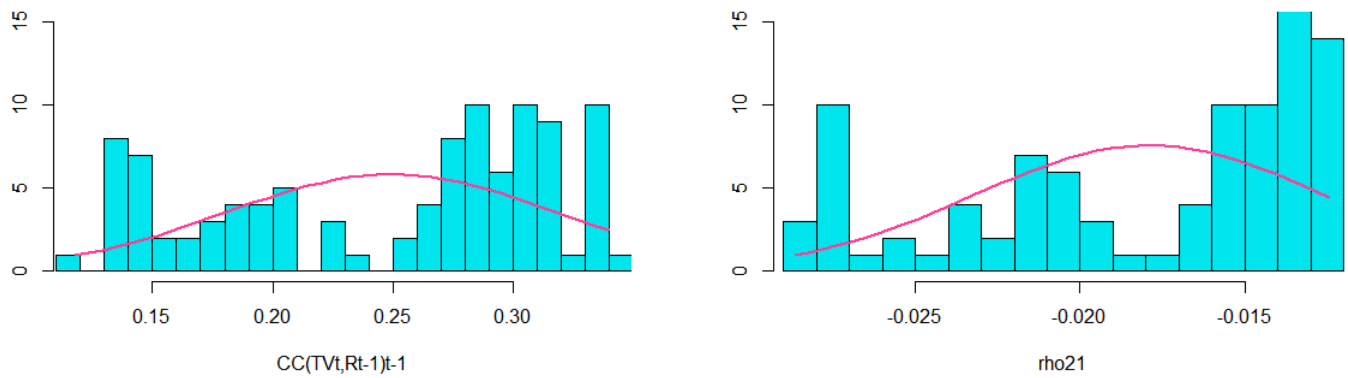

Figure B.3: Histograms of cross-correlations of tv_t^S, r_{t-1}^S and of $\hat{\rho}_{21t}$

Figure B.3 compares the histograms of cross-correlations of tv_t^S, r_{t-1}^S and the estimated $\hat{\rho}_{21t}$ from equation (2.10). In both plots the red line shows the kernel density estimates.

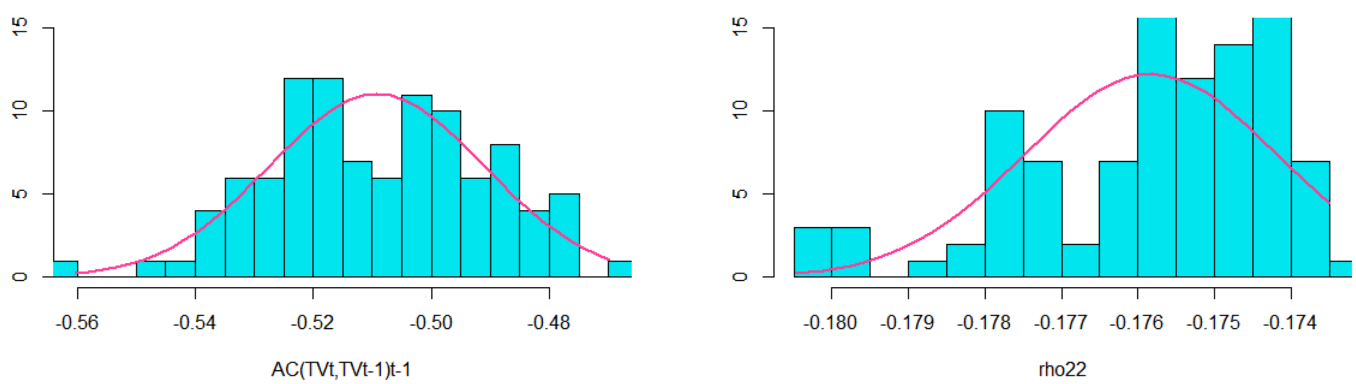

Figure B.4: Histograms of autocorrelations at lag one of tv_t^S and of $\hat{\rho}_{22t}$

Figure B.4 compares the histograms of autocorrelation at lag one of SPDR's trade volume changes and the estimated $\hat{\rho}_{22t}$ from equation (2.10). In both plots the red line shows the kernel density estimates.

B Factor models-scatters and regression lines

The following Figures 25-28 illustrate the regressions of $\hat{\rho}_{jk,t}$ on auto- and cross-correlations of SPDR returns and volume changes (equations 3.13 to 3.16). We observe that the scatters are irregular and the linear models provide fairly good approximations.

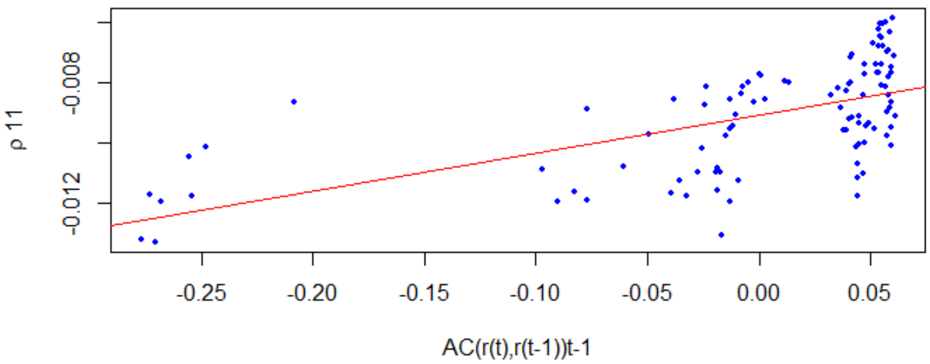

Figure 28: Regression of $\hat{\rho}_{11t}$ on $AC(r^S)_{t-1}$

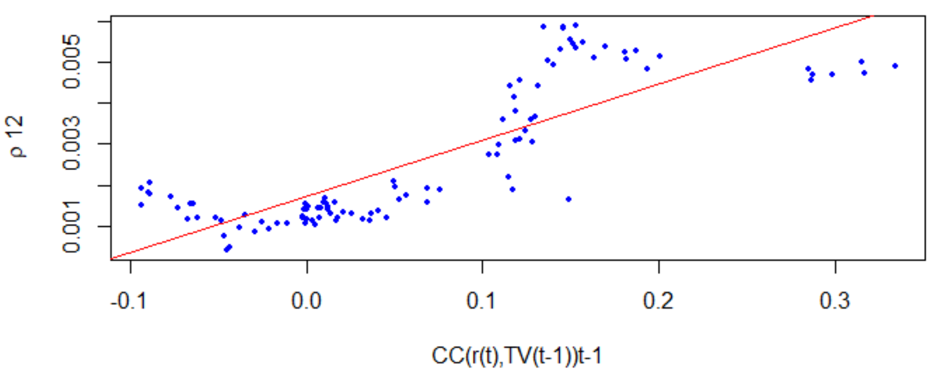

Figure 29: Regression of $\hat{\rho}_{12t}$ on $CC(r^s tv^S)_{t-1}$

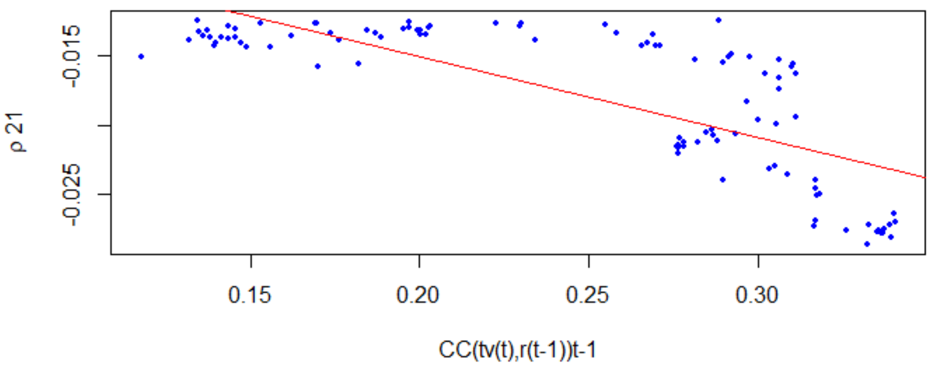

Figure 30: Regression of $\hat{\rho}_{21t}$ on $CC(tv^S r^s)_{t-1}$

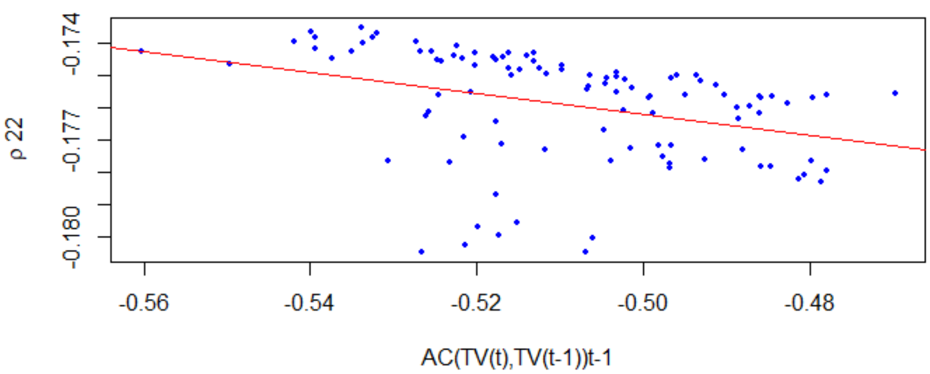

Figure 31: Regression of $\hat{\rho}_{22t}$ on $AC(tv^S)_{t-1}$

C Stochastic autoregressive coefficients

This section illustrates the changes to the optimal allocation vectors when the autoregressive coefficients $\rho_{ij,t+1}$ are considered as random functions of factor F_t. The factor F_t represents jointly the returns r_t^S and trade volume changes tv_t^S of SPDR at time t that determine the autoregressive coefficients $\rho_{ij,t+1}$.

The expected positional utilities to be maximized are as follows:

$$\begin{aligned}
&- E[exp(-\mathscr{A}_r Q^r_{t+1}(\beta'_r r_{t+1})) \mid \underline{r_t}, \underline{tv_t}, \underline{F_t}] \\
&= -E\{E[exp(-\mathscr{A}_r Q^r_{t+1}(\beta'_r r_{t+1})) \mid \underline{r_t}, \underline{tv_t}, \underline{F_{t+1}}] | \underline{r_t}, \underline{tv_t}, \underline{F_t}\} \\
&= -E\Big[exp\Big(- \mathscr{A}_r \rho_{11,t+1} \sum_{i=1}^{n} \beta_{r,i} u_{i,t} - \mathscr{A}_r \rho_{12,t+1} \sum_{i=1}^{n} \beta_{r,i} v_{i,t} + \frac{1}{2} \mathscr{A}_r^2 \sum_{i=1}^{n} \beta^2_{r,i} \sigma^2_{1,t+1}\Big) | \underline{r_t}, \underline{tv_t}, \underline{F_t}\Big] \\
&= -E_t(exp[-\mathscr{A}_r \rho_{11,t+1} \beta'_r u_t - \mathscr{A}_r \rho_{12,t+1} \beta'_r v_t] + \frac{1}{2} \mathscr{A}_r^2 \beta'_r \beta_r \sigma^2_{1,t+1})
\end{aligned} \tag{C.1}$$

subject to $\beta'_r h = 1$ and,

$$\begin{aligned}
&- E[exp(-\mathscr{A}_{tv} Q^{tv}_{t+1}(\beta'_{tv} tv_{t+1})) \mid \underline{r_t}, \underline{tv_t}, \underline{F_t}] \\
&= -E\{E[exp(-\mathscr{A}_{tv} Q^{tv}_{t+1}(\beta'_{tv} tv_{t+1})) \mid \underline{r_t}, \underline{tv_t}, \underline{F_t}] | \underline{r_t}, \underline{tv_t}, \underline{F_t}\} \\
&= -E\Big[exp\Big(-\mathscr{A}_{tv} \rho_{21,t+1} \sum_{i=1}^{n} \beta_{tv,i} u_{i,t} - \mathscr{A}_{tv} \rho_{22,t+1} \sum_{i=1}^{n} \beta_{tv,i} v_{i,t} + \frac{1}{2} \mathscr{A}_{tv}^2 \sum_{i=1}^{n} \beta^2_{tv,i} \sigma^2_{2,t+1}\Big) | \underline{r_t}, \underline{tv_t}, \underline{F_t}\Big] \\
&= -E_t(exp[-\mathscr{A}_{tv} \rho_{21,t+1} \beta'_{tv} u_t - \mathscr{A}_{tv} \rho_{22,t+1} \beta'_{tv} v_t] + \frac{1}{2} \mathscr{A}_r^2 \beta'_{tv} \beta_{tv} \sigma^2_{2,t+1})
\end{aligned} \tag{C.2}$$

subject to $\beta'_{tv} h = 1$.

The above optimization problems are difficult to solve. In order to simplify the optimal allocation vectors, we can consider their first-order expansion with respect to $\rho_{jk,t+1}$ (where $j, k = 1, 2$) for small $\rho_{jk,t+1}$. At first-order approximation with respect to the persistence parameters, we have $E_t(\sigma^2_{1,t+1}) = E_t(\sigma^2_{2,t+1}) \simeq 1$ [14].

[14] In Section 2 we showed that in practice, the positional persistence values at different dates can be rather small (see Figures 1,2). Therefore the assumption that $E_t \sigma^2_{1,t+1} = E_t \sigma^2_{2,t+1} \simeq 1$ is plausible

$$
\begin{aligned}
&- E_t[(1 - \mathscr{A}_r \rho_{11,t+1} \beta_r' u_t - \mathscr{A}_r \rho_{12,t+1} \beta_r' v_t) \exp \frac{1}{2} \mathscr{A}_r^2 \beta_r' \beta_r] \\
&\simeq -(1 - \mathscr{A}_r E_t \rho_{11,t+1} \beta_r' u_t - \mathscr{A}_r E_t \rho_{12,t+1} \beta_r' v_t) \exp \frac{1}{2} \mathscr{A}_r^2 \beta_r' \beta_r \\
&\simeq - \exp[- \mathscr{A}_r E_t \rho_{11,t+1} \beta_r' u_t - \mathscr{A}_r E_t \rho_{12,t+1} \beta_r' v_t - \frac{1}{2} \mathscr{A}_r^2 \beta_r' \beta_r
\end{aligned} \tag{C.3}
$$

$$
\begin{aligned}
&- E_t[(1 - \mathscr{A}_{tv} \rho_{21,t+1} \beta_{tv}' u_t - \mathscr{A}_{tv} \rho_{22,t+1} \beta_{tv}' v_t) \exp \frac{1}{2} \mathscr{A}_{tv}^2 \beta_{tv}' \beta_{tv}] \\
&\simeq -(1 - \mathscr{A}_{tv} E_t \rho_{21,t+1} \beta_{tv}' u_t - \mathscr{A}_{tv} E_t \rho_{22,t+1} \beta_{tv}' v_t) \exp \frac{1}{2} \mathscr{A}_{tv}^2 \beta_{tv}' \beta_{tv} \\
&\simeq - \exp[- \mathscr{A}_{tv} E_t \rho_{21,t+1} \beta_{tv}' u_t - \mathscr{A}_{tv} E_t \rho_{22,t+1} \beta_{tv}' v_t - \frac{1}{2} \mathscr{A}_{tv}^2 \beta_{tv}' \beta_{tv}
\end{aligned} \tag{C.4}
$$

which are objective functions similar to those in Section () with the autoregressive coefficients $\rho_{jk,t+1}$, $j,k = 1,2$ replaced by their expectations $E_t \rho_{jk,t+1}$, $j,k = 1,2$.

Hence, the approximate optimal positional allocations are as follows:

$$
\beta^*_{r,it} = \frac{1}{n} + \frac{1}{\mathscr{A}_r} \Big[E_t \rho_{11,t+1} u_{it} + E_t \rho_{12,t+1} v_{it} - \frac{1}{n} \sum_{i=1}^{n} (E_t \rho_{11,t+1} u_{it} + E_t \rho_{12,t+1} v_{it}) \Big] \tag{C.5}
$$

$$
\beta^*_{tv,it} = \frac{1}{n} + \frac{1}{\mathscr{A}_{tv}} \Big[E_t \rho_{21,t+1} u_{it} + E_t \rho_{22,t+1} v_{it} - \frac{1}{n} \sum_{i=1}^{n} (E_t \rho_{21,t+1} u_{it} + E_t \rho_{22,t+1} v_{it}) \Big] \tag{C.6}
$$

By simplifying the above expressions we get:

$$
\beta^*_{r,it} = \frac{1}{n} + \frac{1}{\mathscr{A}_r} \Big(E_t \rho_{11,t+1} (u_{it} - \overline{u_t}) + E_t \rho_{12,t+1} (v_{it} - \overline{v_t}) \Big) \tag{C.7}
$$

$$
\beta^*_{tv,it} = \frac{1}{n} + \frac{1}{\mathscr{A}_{tv}} \Big(E_t \rho_{21,t+1} (u_{it} - \overline{u_t}) + E_t \rho_{22,t+1} (v_{it} - \overline{v_t}) \Big) \tag{C.8}
$$

where $\overline{u_t} = 1/n\sum_{i=1}^{n} u_{it}$ and $\overline{v_t} = 1/n\sum_{i=1}^{n} v_{it}$ are the cross-sectional averages of the Gaussian ranks at time t. When the number of assets (n) tends to infinity, these cross-sectional averages tend to zero, which is the mean of the standard Normal distribution.

The above optimal allocations are linear combination of two portfolios. The first one has positive weights $\frac{1}{n}$. The second portfolio on the right hand side of each solution is an arbitrage portfolio (zero-cost portfolio), with weights involving the ranks ($E_t\rho_{11,t}u_{it} + E_t\rho_{12,t}v_{it}$ and $E_t\rho_{21,t}u_{it} + E_t\rho_{22,t}v_{it}$, respectively.

D Square root of matrix Σ

To find the matrix $\Sigma^{1/2}$ let us consider:

$$\Sigma = \begin{pmatrix} 1-\rho_{11}^2-\rho_{12}^2 & 1-\rho_{11}\rho_{21}-\rho_{12}\rho_{22} \\ 1-\rho_{21}\rho_{11}-\rho_{22}\rho_{12} & 1-\rho_{21}^2-\rho_{22}^2 \end{pmatrix} = \begin{pmatrix} A & B \\ B & D \end{pmatrix} \tag{A.1}$$

The square root of variance matrix is:

$$\Sigma^{1/2} = \pm\Big(\frac{1}{R}\Big)\begin{pmatrix} A+T & B \\ B & D+T \end{pmatrix} \tag{A.2}$$

where $T = |Det|^{1/2}$ and $R^2 = A + D + 2T$. We get:

$$\begin{aligned} T &= \sqrt{AD-B^2} \\ R &= \sqrt{A+D+2\sqrt{AD-B^2}} \end{aligned} \tag{A.3}$$

By substituting $A.3$ into $A.2$ we get:

$$\begin{aligned} \Sigma^{1/2} &= \pm\left(\frac{1}{\sqrt{A+D+2\sqrt{AD-B^2}}}\right)\begin{pmatrix} A+\sqrt{AD-B^2} & B \\ B & D+\sqrt{AD-B^2} \end{pmatrix} \\ \Sigma^{1/2} &= \pm\begin{pmatrix} \frac{A+\sqrt{AD-B^2}}{\sqrt{A+D+2\sqrt{AD-B^2}}} & \frac{B}{\sqrt{A+D+2\sqrt{AD-B^2}}} \\ \frac{C}{\sqrt{A+D+2\sqrt{AD-B^2}}} & \frac{D+\sqrt{AD-B^2}}{\sqrt{A+D+2\sqrt{AD-B^2}}} \end{pmatrix} \end{aligned} \tag{A.4}$$

By substituting A, B and D from equation $A.1$ into equation $A.4$ we get the following:

$$
\begin{aligned}
T &= \sqrt{\rho_{21}^2(-1+\rho_{12}^2)+\rho_{22}^2(-1+\rho_{11}^2)-\rho_{11}^2-\rho_{12}^2+2\rho_{11}\rho_{21}(1-\rho_{22}\rho_{12})+2\rho_{22}\rho_{12}} \\
R &= \sqrt{2-(\rho_{11}^2+\rho_{12}^2+\rho_{21}^2+\rho_{22}^2)+2\sqrt{T}}
\end{aligned}
\tag{A.5}
$$

$$
\Sigma^{1/2} = \pm \begin{pmatrix} \frac{1-\rho_{11}^2-\rho_{12}^2+\sqrt{T}}{\sqrt{2-(\rho_{11}^2+\rho_{12}^2+\rho_{21}^2+\rho_{22}^2)+2\sqrt{T}}} & \frac{1-\rho_{11}\rho_{21}-\rho_{12}\rho_{22}}{\sqrt{2-(\rho_{11}^2+\rho_{12}^2+\rho_{21}^2+\rho_{22}^2)+2\sqrt{T}}} \\ \frac{1-\rho_{21}\rho_{11}-\rho_{22}\rho_{12}}{\sqrt{2-(\rho_{11}^2+\rho_{12}^2+\rho_{21}^2+\rho_{22}^2)+2\sqrt{T}}} & \frac{1-\rho_{21}^2-\rho_{22}^2+\sqrt{T}}{\sqrt{2-(\rho_{11}^2+\rho_{12}^2+\rho_{21}^2+\rho_{22}^2)+2\sqrt{T}}} \end{pmatrix}
\tag{A.6}
$$