Appendix 2

Condition for distinguishing a mixed process from a pure process, $n = 2$

As it is equivalent to identify process (Y_t), or a one-to-one linear transformation of (Y_t), we assume that the mixed causal/noncausal process is (Y_t^*) itself, with the autocovariances given in (3.3)-(3.4).

By Proposition 3, we consider a pure causal process, without loss of generality. Let us denote such a process by : $Y_t = \Phi Y_{t-1} + \varepsilon_t$, where the eigenvalues of Φ are of modulus strictly smaller than 1, and analyze the conditions ensuring that the associated autocovariances $\Gamma(h)$ coincide with the autocovariances given in (3.3)-(3.4). Let us assume for expository purpose that $J_1 \neq 1/J_2$. Then matrix Φ' is diagonalizable with eigenvalues J_1 and $1/J_2$, and we can write :

$$\Phi' = C^{-1} \begin{pmatrix} J_1 & 0 \\ 0 & J_2^{-1} \end{pmatrix} C,$$

where $C = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}$ is the matrix, whose columns provide the eigenvectors of Φ'. For $h \leq 0$, we have from (3.2):

$$\Gamma(h) = \Gamma(0)(\Phi')^{|h|} = \Gamma(0)C^{-1} \begin{pmatrix} J_1^{|h|} & 0 \\ 0 & J_2^{-|h|} \end{pmatrix} C.$$

Let us now consider the implications of the conditions $\gamma_{1,2}^*(h) = 0$, if $h \leq 0$. We get :

$$(1,0)\Gamma(h) \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0, \forall h, h \leq 0,$$

$$\Leftrightarrow \quad (1,0)\Gamma(0)C^{-1} \begin{pmatrix} J_1^{|h|} & 0 \\ 0 & J_2^{-|h|} \end{pmatrix} C \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0, \forall h \leq 0,$$

$$\Leftrightarrow \quad (d_{11}, d_{12}) \begin{pmatrix} J_1^{|h|} & 0 \\ 0 & J_2^{-|h|} \end{pmatrix} \begin{pmatrix} c_{12} \\ c_{22} \end{pmatrix} = 0, \forall h \leq 0,$$

(where $(d_{11}, d_{12}) = (1,0)\Gamma(0)C^{-1}$)

$\Leftrightarrow d_{11}c_{12}J_1^{|h|} + d_{12}c_{22}J_2^{-|h|} = 0, \forall h \leq 0,$

$\Leftrightarrow$ either $c_{12} = 0$ and $d_{12} = 0$, or $c_{22} = 0$ and $d_{11} = 0$,

(because vectors $(d_{11}, d_{12})'$ and $(c_{12}, c_{22})'$ are non-zero vectors and the sequences $J_1^{|h|}$ and $J_2^{-|h|}$ are linearly independent).

a) Let us consider the case $c_{12} = 0$. Then c_{22} can be standardized to 1.

We have :

$C = \begin{pmatrix} c_{11} & 0 \\ c_{21} & 1 \end{pmatrix}, \ C^{-1} = \frac{1}{c_{11}} \begin{pmatrix} 1 & 0 \\ -c_{21} & c_{11} \end{pmatrix},$

$(1,0)\Gamma(0)C^{-1} = \frac{1}{c_{11}}(\gamma_{11}(0), \gamma_{12}(0)) \begin{pmatrix} 1 & 0 \\ -c_{21} & c_{11} \end{pmatrix},$

and $d_{12} = \gamma_{12}(0)$.

Thus the condition $d_{12} = 0$ is equivalent to the condition $\gamma_{12}(0) = 0$.

b) Similarly, if $c_{22} = 0$, we can fix $c_{12} = 1$. We have :

$$C = \begin{pmatrix} c_{11} & 1 \\ c_{21} & 0 \end{pmatrix}, C^{-1} = -\frac{1}{c_{21}} \begin{pmatrix} 0 & -1 \\ -c_{21} & c_{11} \end{pmatrix},$$

$$(1,0)\Gamma(0)C^{-1} = -\frac{1}{c_{21}}[\gamma_{11}(0), \gamma_{12}(0)] \begin{pmatrix} 0 & -1 \\ -c_{21} & c_{11} \end{pmatrix},$$

and $d_{11} = \gamma_{12}(0)$. The condition $d_{11} = 0$ is equivalent to the condition $\gamma_{12}(0) = 0$.