Nonlinear Innovations and Impulse Responses

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Abstract

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This paper introduces an impulse response analysis of nonlinear processes based on the concept of nonlinear innovation. Our approach borrows from the traditional linear impulse response analysis in that we consider shocks to innovations of a process. It also extends the methods of nonlinear impulse response analysis proposed earlier in the literature, in that it eliminates the problem of serial correlation of error terms, allows to examine permanent shocks, i.e. shocks occurring repeatedly in time, and ensures a straightforward interpretation of transitory, symmetric shocks. It can be applied to processes such as the popular GARCH, or ACD. As well it can be used for shock sensitivity evaluation of dynamic financial strategies. As an illustration we use impulse responses to determine the Value at Risk and the minimum capital requirement under a dynamic portfolio management.

Keywords: Nonlinear Dynamics, Gaussian Innovations, Volterra Expansion, Impulse Response, ACD Model, Value at Risk.

JEL : C10, C22, C52, G11

Résumé

Innovations non-linéaires et fonctions réponses

Nous introduisons un concept d'innovation adapté à l'analyse des dynamiques non linéaires. Nous expliquons comment le processus initial peut être exprimé en fonction des valeurs présentes et passées de l'innovation, utilisons les résidus associés pour construire des tests de spécification d'une dynamique non linéaire et pour définir des fonctions réponses à des chocs transitoires ou permanents.

Mots clés: Dynamiques non linéaires, innovation gaussienne, développement de Volterra, fonction réponse, modèle ACD, valeur à risque.

JEL : C10, C22, C52, G11

1 Introduction

The impulse response analysis is a widely used technique for exploring the dynamics of ARIMA models. It consists in tracing out and examining the response pattern of a model to external shocks. This pattern is called an impulse response function and formally defined in the literature as the time profile of the shock effect on the behavior of the series [see, e.g. Koop, Pesaran, Potter (1996)]. In recent literature, Gallant, Rossi, Tauchen (1993), and Koop, Pesaran, Potter (1996), [henceforth GRT and KPP] have pointed out essential differences between the impulse response analysis of linear and nonlinear processes. They can be briefly summarized as follows.

i) In ARIMA models, the impulse responses feature the property of symmetry (i.e. a transitory shock of $-\delta$ has exactly the opposite effect of a transitory shock of $+\delta$) , whereas for nonlinear processes the effects of shocks of opposite signs may be very different.

ii) The property of "shock linearity" [i.e. a transitory shock of $k\delta$ has k times the effect of a shock of δ] is satisfied in the linear framework whereas otherwise the effect of the magnitude of a shock is nonlinear.

iii) In ARIMA models the shock effect does not depend on the history of the process, while the path dependency is crucial in nonlinear framework.

iv) In nonlinear framework, it is necessary to consider the distributional properties of the impulse response function, besides tracing out the effect of a shock to the conditional expectation of the future variable of interest.

GRT and KPP have proposed extensions of the traditional impulse response analysis to nonlinear dynamic models. A common feature of these two approaches is their focus on impulse responses to transitory shocks, leaving out of scope the permanent shock analysis. Substantial differences between them are due to the adopted techniques. The analysis by GRT is performed conditionally on the observed history of the process, whereas KPP propose to integrate out the possible histories. As well the transitory shocks are differently defined. In the spirit of the Keynesian multiplier analysis, GRT propose to shock directly the variable, instead of the innovation. The drawback of this approach is twofold. It can't be extended to an analysis of permanent shocks, i.e. shocks occurring repeatedly over an infinite time horizon. As well the concept of symmetric shocks is unclear, since symmetric shocks to innovations do not necessarily correspond to symmetric shocks to the current variable. On the contrary, KPP following Sims (1972) [see also Blanchard, Quah (1989)], consider shocks to innovations. In particular, they study innovations obtained by conditionally centering and rescaling the variable Y_t. These innovations:

$$v_t = (V_{t-1}Y_t)^{-\frac{1}{2}}(Y_t - E_{t-1}Y_t),$$

may not be sufficiently corrected for nonlinear temporal dependence. Indeed they may feature temporal dependence in moments of order larger then two, affecting therefore the interpretation of impulse responses. As well the conditional distribution of v_t is, in general, asymmetric. In consequence, the interpretation of symmetric shocks is impeded again.

The aim of the present paper is to improve upon the GRT and KPP methods in these respects. Essentially, problems in extending the linear impulse response analysis to nonlinear framework are due to an ambiguous definition of the innovation of a nonlinear process. Indeed, in the time series literature, innovations of a stochastic process (Y_t) are defined either as a) errors representing differences between the expected and realized values of Y_t, i.e. $\epsilon_t^{(1)} = Y_t - E_{t-1}Y_t$, or b) conditionally standardized expectation errors, i.e. $\epsilon_t^{(2)} = (Y_t - E_{t-1}Y_t)/(V_{t-1}Y_t)^{\frac{1}{2}}$, where $E_{t-1}Y_t$ and $V_{t-1}Y_t$ are the conditional mean and variance of Y_t given the information available at time $t-1$. These definitions are not well suited for the analysis of nonlinear dynamics for the following reasons. The standardized innovations $(\epsilon_t^{(2)})$ may not necessarily be independent, due to unobserved cross effects of their conditional moments of order strictly larger than two. Moreover, in a nonlinear framework, we are not only interested in the prediction of the variable Y_t itself, but also in the prediction of some nonlinear transformations of the variable. However the definitions of innovations $\epsilon^{(1)}$ and $\epsilon^{(2)}$ are inadequate in this context. For example, $\epsilon_t^{(2)} = (Y_t - E_{t-1}Y_t)/(V_{t-1}Y_t)^{\frac{1}{2}}$ and $\epsilon_t^{(2)}(\exp) = (\exp Y_t - E_{t-1}\exp Y_t)/V_{t-1}(\exp Y_t)$ do not generally satisfy a one-to-one relationship. Thus it is not possible to assimilate a shock to $\epsilon^{(2)}$ with a shock to $\epsilon_t^{(2)}(\exp)$. In the present paper we introduce a notion of innovation for nonlinear processes which eliminates this ambiguity. Our response analysis borrows from the traditional approach in that it considers shocks to innovations. Its novelty consists in using the proposed nonlinear innovations, instead ot the traditional ones to trace impulse responses. As an illustration, we extend the domain of application of the impulse response analysis to financial strategies, and consider shock effects to an outcome of a dynamic strategy of portfolio hedging.

In Section 2 we define a nonlinear Gaussian innovation of a strongly stationary process and derive a representation theorem for nonlinearly regular stationary process where the current value of the process is expressed as a nonlinear function of current and lagged nonlinear innovations. Section 3 concerns nonlinear impulse response analysis. We study the effects of permanent and transitory shocks to nonlinear Gaussian innovations and compare our approach to impulse response techniques introduced by Gallant, Rossi, Tauchen (1993), and Koop, Pesaran, Potter (1996). Section 4 extends the impulse response techniques to a setup involving dynamic financial strategies. As an illustration we discuss the structural impulse response analysis for determining the Value at Risk and the minimum capital requirement under a dynamic portfolio management. Section 5 concludes the paper.

2 Nonlinear Gaussian innovation and representation theorem

In this section we define the nonlinear innovations and propose a representation theorem for nonlinear time series. The section also provides examples of nonlinear innovations for processes from the class of Nonlinear ARMA (NLARMA) processes.

We consider an unidimensional strongly stationary process $(Y_t, t \in Z)$, and denote by $\mathcal{F}_t = \sigma(\underline{Y_t})$ the sigma algebra generated by the current and past values of the process, where $\underline{Y_t} = Y_t, Y_{t-1}, Y_{t-2}...$. Moreover we assume that the distribution of the current realization of the process conditioned on its past is continuous.

Assumption A.1: The conditional distribution of Y_t given $\mathcal{F}_{t-1}$ is continuous on $[\mathcal{R}, \mathcal{B}(\mathcal{R})]$ with a positive p.d.f. denoted by f_{t-1}.

The conditional c.d.f. denoted F_{t-1} is continuous, strictly increasing and hence invertible.

2.1 Nonlinear Gaussian innovations

Intuitively, a nonlinear innovation (η_t) of the process (Y_t) is a strong white noise, such that the current value Y_t can be written as a function of its own past and the current value η_t of the innovation process: $Y_t = g(\eta_t, \underline{Y_{t-1}})$, say, and Y_t and η_t satisfy a one to one relationship for a given $\underline{Y_{t-1}}$ [1]. Hence, if a nonlinear innovation exists, it is not uniquely defined, since any one to one transformation of a nonlinear innovation is also a nonlinear innovation. In this subsection, we prove the existence of a nonlinear innovation and show that the multiplicity of nonlinear innovations can be eliminated by introducing the identifiability condition of a Gaussian white noise.

Definition 1: The process $(\epsilon_t, t \in Z)$ is a nonlinear Gaussian innovation of the process $(Y_t, t \in Z)$ if it satisfies the following conditions:

i) $(\epsilon_t, t \in Z)$ is a Gaussian white noise IIN(0,1);

ii) ϵ_t and Y_t satisfy a continuous invertible relationship conditional on $\mathcal{F}_{t-1}$: $\epsilon_t = g_{t-1}(Y_t)$ a.s., where g_{t-1} is continuous, invertible and may depend on the past $\mathcal{F}_{t-1}$.

Property 2: Under Assumption A.1, the strongly stationary process $(Y_t, t \in Z)$ admits a nonlinear Gaussian innovation. It is unique up to a change of sign, date by date.

Proof:

[1] It is important to note that generally the standard innovations $\epsilon_t^{(1)}$ and $\epsilon_t^{(2)}$ presented in the introduction are not nonlinear innovations in this sense, since they do not satisfy the strong white noise property, and a one to one relationship with Y_t for a given $\underline{Y_{t-1}}$.

i) The process $(\epsilon_t, t \in Z)$ defined by:

$$\epsilon_t = \Phi^{-1}[F_{t-1}(Y_t)], \quad t \in Z, \tag{2.1}$$

where Φ denotes the c.d.f. of the standard Normal distribution, satisfies the two conditions of definition 1.

ii) Let us assume that ϵ_t^1 is another nonlinear Gaussian innovation. Then there exists a continuous invertible relationship between ϵ_t and ϵ_t^1 conditional on $\mathcal{F}_{t-1} : \epsilon_t = h_{t-1}(\epsilon_t^1)$, (say), and
$P[\epsilon_t^1 < \epsilon] = \Phi(\epsilon) = P[h_{t-1}(\epsilon_t^1) < \epsilon \mid \mathcal{F}_{t-1}]$.

The function h_{t-1} is continuous, invertible on R and therefore monotone. If it is increasing, conditional on $\mathcal{F}_{t-1}$, we get:

$$\Phi(\epsilon) = \Phi[h_{t-1}^{-1}(\epsilon)], \;\; \forall \epsilon,$$

which implies $h_{t-1} = Id$.

If it is decreasing, conditional on $\mathcal{F}_{t-1}$, we get:

$$\Phi(\epsilon) = 1 - \Phi[h_{t-1}^{-1}(\epsilon)], \;\; \forall \epsilon,$$

which implies $h_{t-1} = -Id$.

Q.E.D.

Therefore it is always possible to chose a Gaussian innovation process satisfying an increasing relationship between Y_t and ϵ_t at any time t. The corresponding innovation is uniquely defined by (2.1). Moreover this formula implies that the gaussian nonlinear innovations of (Y_t) and of an invertible increasing transform of (Y_t) are identical. The choice of a Gaussian distribution is purely conventional, but has several advantages. Firstly it is appropriate for deriving the Volterra representation of a nonlinear time series; secondly the distribution of ϵ_t is symmetric, which is useful for examining symmetric shocks.

2.2 Representation theorem

In this section we give a representation theorem which expresses the current value of the process as a function of current and lagged values of innovations. By analogy to the Wold representation of linear processes, we first introduce the nonlinear regularity condition.

Definition 3: The $(Y_t, t \in Z)$ process is nonlinearly regular if $\mathcal{F}_{-\infty} = \cap_t \mathcal{F}_t$ is empty.

This regularity condition implies in particular that [Granger (1995)] [2]:

$$\lim_{h\to\infty} E(a(Y_t, Y_{t+1}, ..., Y_{t+q})| \mathcal{F}_{t-h}) = Ea(Y_t, Y_{t+1}, ..., Y_{t+q}),$$

for any integer q and an integrable function a, for whom the limiting expectation exists. This means that the initial value of the process does not help to predict over a long horizon. In the following property, the underlined variables denote again the realizations up to and including the given date, i.e. $\underline{\epsilon_t} = \epsilon_t, \epsilon_{t-1}, \epsilon_{t-2}, ...,$ and $\underline{Y_t}$ is defined analogously.

Property 4: If the strongly stationary process $(Y_t, t \in Z)$ satisfies assumption A.1 and is nonlinearly regular, then $\sigma(\underline{\epsilon_t}) = \sigma(\underline{Y_t}) = \mathcal{F}_t$.

Proof:

The second condition of definition 1 implies:

$$\sigma(\epsilon_t, \mathcal{F}_{t-1}) = \sigma(Y_t, \mathcal{F}_{t-1}), \quad \forall t, \tag{2.2}$$

and by recursion:

$$\sigma(\epsilon_t, \epsilon_{t-1}, ..., \epsilon_{t-p+1}, \mathcal{F}_{t-p}) = \sigma(Y_t, Y_{t-1}, ..., Y_{t-p+1}, \mathcal{F}_{t-p}), \quad \forall t,\ p \geq 0. \tag{2.3}$$

i) We have: $\sigma(\epsilon_t, ..., \epsilon_{t-p}) \subset \sigma(\underline{Y_t}),\ \ \forall p$, and then $\sigma(\underline{\epsilon_t}) = \vee_p\, \sigma(\epsilon_t, ..., \epsilon_{t-p}) \subset \sigma(\underline{Y_t})$.

ii) Conversely:

$$\sigma(\underline{Y_t}) = \sigma(\epsilon_t, ..., \epsilon_{t-p}) \vee \mathcal{F}_{t-p} \subset \sigma(\underline{\epsilon_t}) \vee \mathcal{F}_{t-p}, \ \ \forall p.$$

Therefore $\sigma(\underline{Y_t}) \subset \cap_p\, [\sigma(\underline{\epsilon_t}) \vee \mathcal{F}_{t-p}] = \sigma(\underline{\epsilon_t}) \vee (\cap_p \mathcal{F}_{t-p}) = \sigma(\underline{\epsilon_t})$, due to the regularity condition. Q.E.D.

The representation theorem is a consequence of the existence of a simple Hilbertian basis for Gaussian processes. More precisely, let us introduce the Hermite polynomials:

$$H_j(\epsilon) = \sum_{0 \leq m \leq [j/2]} \frac{j!}{(j-2m)!m!2^m} (-1)^m \epsilon^{j-2m} \quad j = 0, 1, ... \tag{2.4}$$

A Hilbertian basis of $L^2(\sigma(\underline{\epsilon_t}))$ is given by:

$$\frac{1}{\sqrt{j_1!}} H_{j_1}(\epsilon_{t-h_1}) \frac{1}{\sqrt{j_2!}} H_{j_2}(\epsilon_{t-h_2}) ... \frac{1}{\sqrt{j_n!}} H_{j_n}(\epsilon_{t-h_n}), \tag{2.5}$$

$n, j_1, ..., j_n, h_1, ..., h_n$ varying with $h_1 \neq h_2 .... \neq h_n$.

The representation theorem follows directly.

[2] It means that the process has short memory in the mean for any nonlinear transformation, using Granger's terminology [Granger (1995)].

Property 5: If the strongly stationary process satisfies assumption A.1, is nonlinearly regular and square integrable, we get:

$$Y_t = \lim_{\substack{N \to \infty \\ J \to \infty \\ H \to \infty}} \sum_{n=1}^{N} \sum_{j_1,\dots,j_n=1,\dots,J} \sum_{\substack{h_1,\dots,h_n=0,\dots,H \\ h_1 \neq \dots \neq h_n}} a^{(N,J,H)}_{j_1,\dots,j_n,h_1,\dots,h_n} H_{j_1}(\epsilon_{t-h_1}) H_{j_2}(\epsilon_{t-h_2}) \dots H_{j_n}(\epsilon_{t-h_n}),$$

where Y_t is the mean square limit.

This representation theorem is of Volterra type [see, Volterra (1930), (1959), Nisio (1960), Priestley (1988)] and defines Y_t as a limit of polynomials in current and lagged values of a Gaussian white noise. The traditional derivation of the Volterra expansion is based on Fourrier transform arguments. Our approach concerns the time domain and is closer to the lines followed by Wiener (1958), using Hilbert arguments [3].

The condition of square integrability of Y_t is not very restrictive. Indeed if Y_t is not square integrable we may find an increasing transformation $h(Y_t)$, which will satisfy this requirement. Next, the representation theorem can be applied to the process $h(Y_t)$ and by inverting h, a representation for Y_t will be obtained [where Y_t becomes now the limit in probability].

2.3 Nonlinear ARMA Processes

The nonlinear innovations can be used to define a class of Nonlinear ARMA (NLARMA) models.

Definition 7: The strongly stationary process $(Y_t, t \in Z)$ has a nonlinear ARMA(p,q) or NLARMA(p,q) representation if and only if it satisfies a recursive relation [see Granger, Terasvirta (1993)]:

$$Y_t = g(Y_{t-1}, \dots, Y_{t-p}, \epsilon_t, \dots, \epsilon_{t-q}), \tag{2.7}$$

where $(\epsilon_t, t \in Z)$ is a Gaussian nonlinear innovation, g is a function which is invertible with respect to ϵ_t, and is not constant with respect to Y_{t-p}, ϵ_{t-q} respectively. The coefficients p and q are the (nonlinear) autoregressive and moving average orders, respectively.

i) Nonlinear Autoregressive Process

For $q = 0$, we get a nonlinear autoregression (NLAR) of order p [see Tong (1990), p 96]:

$$Y_t = g(Y_{t-1}, \dots, Y_{t-p}, \epsilon_t) \Leftrightarrow c(Y_t, Y_{t-1}, \dots, Y_{t-p}) = \epsilon_t, \quad \text{say}.$$

[3] The representation has an especially simple form, when the coefficients $a^{(N,J,H)}_{j_1,\dots,j_n,h_1,\dots,h_n}$ are independent of N, J, H. Indeed we get:

$$Y_t = \sum_{n=1}^{\infty} \sum_{j_1,\dots j_n=0}^{\infty} \sum_{h_1,\dots,h_n=0}^{\infty} a_{j_1,\dots,j_n,h_1,\dots,h_n} H_{j_1}(\epsilon_{t-h_1}) H_{j_2}(\epsilon_{t-h_2}) \dots H_{j_n}(\epsilon_{t-h_n}). \tag{2.6}$$

It is known that linear Gaussian ARMA models satisfy this condition whenever the moving average part does not admit a root with unitary modulus [Whittle (1963)].

It is easy to check that a strongly stationary process admits a NLAR representation of order p if and only if it is Markov of order p.

Finally note that $\epsilon_t^{(1)}$ [resp. $\epsilon_t^{(2)}$] is a nonlinear innovation if the NLAR model can be written as:

$$Y_t = g(Y_{t-1}, ..., Y_{t-p}) + h(\epsilon_t),$$

$$[\text{resp. } Y_t = g_0(Y_{t-1}, ..., Y_{t-p}) + g_1(Y_{t-1}, ..., Y_{t-p})\, h(\epsilon_t)],$$

where h is a one to one transformation. Thus these standard definitions are valid only for a restricted class of dynamic models.

ii) Nonlinear Moving Average Process

When $p = 0$ we obtain a nonlinear moving average (NLMA) of order q [see Tong (1990), p 115]:

$$Y_t = g(\epsilon_t, \epsilon_{t-1} ..., \epsilon_{t-q}).$$

For a NLMA(q) process, the sigma algebras $\sigma(Y_t)$ and $\mathcal{F}_{t-q-1} = \sigma(\underline{Y_{t-q-1}})$ are independent. Therefore we find zero autocorrelations between nonlinear transforms of current and lagged observations whenever the lag is large enough. This condition extends the common condition $Cov(Y_t, Y_{t-h}) = 0, \ \forall h \geq q+1$ valid in the linear framework. However, contrary to the linear case, it is not sufficient to characterize the nonlinear moving average process due to cross effects.

iii) The ACD Model

We now present an example of a mixed NLARMA(1,1) model.

The traditional nonlinear dynamic models introduced for financial applications such as the ARCH models [Engle (1982)] or the ACD models [Engle, Russell (1998)] usually contain nonlinear innovations in their specifications. As an illustration let us consider the ACD(1,1) model, where the process of interest is a sequence of durations $\{Y_t,\ t \in \mathcal{Z}\}$. Let us introduce the conditional expectation of Y_t given the past: $\Psi_t = E(Y_t | \mathcal{F}_{t-1})$. It is assumed that the standardized durations Y_t/Ψ_t are independent with identical distributions whose c.d.f is F (say), and that Ψ_t satisfies the recursive equation:

$$\Psi_t = c + \alpha Y_{t-1} + \beta \Psi_{t-1}. \tag{2.8}$$

The nonlinear Gaussian innovation is:

$$\epsilon_t = \Phi^{-1} F(Y_t/\Psi_t). \tag{2.9}$$

Therefore we can write:

$$Y_t = \Psi_t g(\epsilon_t), \tag{2.10}$$

where $g = F^{-1} \cdot \Phi$, and by substituting into the recursive equation (2.8) we get:

$$\begin{aligned} Y_t &= cg(\epsilon_t) + \alpha Y_{t-1} g(\epsilon_t) + \beta Y_{t-1} \frac{g(\epsilon_t)}{g(\epsilon_{t-1})} \\ &= \left[\alpha g(\epsilon_t) + \beta \frac{g(\epsilon_t)}{g(\epsilon_{t-1})}\right] Y_{t-1} + cg(\epsilon_t), \end{aligned} \tag{2.11}$$

which is an autoregressive representation with time dependent random autoregressive coefficients.

We may also write the recursive equation (2.5) as:

$$\Psi_t = c + \alpha \Psi_{t-1} g(\epsilon_{t-1}) + \beta \Psi_{t-1}, \tag{2.12}$$

which shows that the process (ϵ_t) is the nonlinear innovation of the expectation process (Ψ_{t+1}) as well. Therefore (ϵ_t) is the common innovation process of the duration series, its expectation and of the duration volatilities.

3 Impulse Response Analysis

This section presents our approach to the impulse response analysis, provides analytical results on various shock dissipation patterns and illustrates the method by simulations.

3.1 The impulse response function

The impulse response function measures the effect of a sequence of shocks to a nonlinear innovation process on future values of the process of interest. Since the nonlinear innovation is not uniquely defined, we expect to find different impulse response functions depending on the selected nonlinear innovation. We first consider the impulse response function for the Gaussian nonlinear innovation. Next we extend this basic concept to a general impulse response function.

i) Gaussian Nonlinear Innovation.

The impulse response analysis can be based on the Volterra type of decomposition (see, Property 5), where:

$$Y_t = a(\epsilon_t, \epsilon_{t-1}, ..., \epsilon_1, \underline{\epsilon_0}), \tag{3.1}$$

(ϵ_t) is a Gaussian white noise, with unitary variance, and $\underline{\epsilon_0} = \epsilon_0, \epsilon_{-1}, ...$. Since the distribution of ϵ_t is symmetric, the shocks of δ and $-\delta$ have the same infinitesimal occurrence. Thus we can compare the effects of shocks with identical occurrences, called symmetric shocks in the literature. As well, since the distribution of ϵ_t is independent of time, the shocks of the same magnitude δ at different dates also have the same infinitesimal occurrence, which allows us to consider the so-called "permanent" shocks.

As suggested by GRT (1993) the analysis needs to be conditioned on the history before the shock arrival. Therefore if the shocks hit the process at date 1, the previous values of the process and innovations are known, i.e. $\underline{\epsilon_0}$ is fixed. Then, at date 0, we have to evaluate the effect of a sequence of deterministic shocks $\delta_1, \delta_2, ..., \delta_t, ...$ occurring at future dates on the future profile of the process. These effects have to be measured with respect to a benchmark which is the path followed in the absence of shocks. Since future innovations are unknown, this benchmark is random. We denote by: $\epsilon_1^s, \epsilon_2^s, ..., \epsilon_t^s, ...$ a future path for the innovations, where $\epsilon_1^s, \epsilon_2^s, ..., \epsilon_t^s, ...$ are IIN (0,1) conditional on $\underline{\epsilon_0} = \epsilon_{-1}, \epsilon_{-2}, ...$ The random benchmark is:

$$Y_t^s(\underline{\epsilon_0}) = a(\epsilon_t^s, \epsilon_{t-1}^s, ..., \epsilon_1^s, \underline{\epsilon_0}), \tag{3.2}$$

whereas the profile after the shock arrival is:

$$Y_t^s(\underline{\delta}, \underline{\epsilon_0}) = a(\epsilon_t^s + \delta_t, \epsilon_{t-1}^s + \delta_{t-1}, ..., \epsilon_1^s + \delta_1, \underline{\epsilon_0}), \tag{3.3}$$

where $\underline{\delta} = (\delta_1, ..., \delta_t, ...)$.

The entire effect of the sequence of shocks, i.e. the **response**, is summarized by the joint path distribution of:

$$[Y_t^s(\underline{\epsilon_0}), , Y_t^s(\underline{\delta}, \underline{\epsilon_0}),\ t \geq 1].$$

In practice various sequences of shocks are examined. The standard impulse response analysis concerns:

· either transitory shocks at date 1: $\delta_1 = \delta, \delta_t = 0, t \geq 2$,

· or transitory shocks at date t_0: $\delta_{t_0} = \delta,\ \delta_t = 0,\ t \neq t_0$,

· or permanent shocks starting at date 1: $\delta_t = \delta,\ \ t \geq 2$.

They differ in terms of the sign and the magnitude of δ.

Remark: It is common in the literature [GRT(1998), KPP(1996)] to summarize the response by considering either the differences of expectations of the series:

$$E[Y_t^s(\underline{\delta}, \underline{\epsilon_0})|\underline{\epsilon_0}] - E[Y_t^s(\underline{\epsilon_0})|\underline{\epsilon_0}],$$

or the differences of expectations of the transformed series:

$$E[g(Y_t^s(\underline{\delta}, \underline{\epsilon_0}))|\underline{\epsilon_0}] - E[g(Y_t^s(\underline{\epsilon_0}))|\underline{\epsilon_0}],$$

where g is a given nonlinear function,

or differences of variances:

$$V[Y_t^s(\underline{\delta}, \underline{\epsilon_0})|\underline{\epsilon_0}] - V[Y_t^s(\underline{\epsilon_0})|\underline{\epsilon_0}].$$

ii) General Nonlinear Innovation

Let us now consider another nonlinear innovation process: $\eta_t = h(\epsilon_t)$, say, where h is an invertible increasing function. We can write:

$$\begin{aligned} Y_t &= b(\eta_t, \eta_{t-1}, ..., \eta_1, \underline{\eta_0}) \\ &= a[h^{-1}(\eta_t), h^{-1}(\eta_{t-1}), ..., h^{-1}(\eta_1), \underline{h^{-1}(\eta_0)}]. \end{aligned}$$

For clarity of exposition, let us consider a transitory shock δ_1^* at date 1 to the innovation η_1. The profile after the shock arrival is:

$$\begin{aligned} Y_t(\delta) &= a[h^{-1}(\eta_t), h^{-1}(\eta_{t-1}), ..., h^{-1}(\eta_1 + \delta_1^*), \underline{h^{-1}(\eta_0)}] \\ &= a[h^{-1}(\eta_t), h^{-1}(\eta_{t-1}), ..., h^{-1}(\eta_1) + h^{-1}(\eta_1 + \delta_1^*) - h^{-1}(\eta_1), \underline{h^{-1}(\eta_0)}], \end{aligned}$$

We note that the deterministic shock to the nonlinear innovation η_1 is identical to a stochastic shock of magnitude: $\delta_1 = h^{-1}(\eta_1 + \delta_1^*) - h^{-1}(\eta_1) = h^{-1}(h(\epsilon_1) + \delta_1^*) - \epsilon_1$, to the gaussian nonlinear innovation ϵ_1. Thus a shock to the innovation can be deterministic or stochastic, depending on the selected nonlinear innovation.

Remark: A similar argument holds for a deterministic drift of the observation Y_1 itself [see, GRT (1993)]. A deterministic drift of Y_1 corresponds generally to a stochastic drift of **any** nonlinear innovation η_1. The deterministic drifts of Y_1 and η_1 are compatible if and only if we can write $Y_1 = b_0(\underline{\eta_0}) + b_1(\underline{\eta_0})\eta_1$.

In fact it is only possible to reconcile the definitions of the shocks and of the innovations if we interpret the impulse as a shock to the distribution of the innovation and not as a shock to the innovation value. In our case of the gaussian nonlinear innovation, an impulse shifts the standard Normal distribution N(0,1) by δ, yielding the distribution N(δ,1). Therefore, it can be viewed as a drift impulse. However other impulses would also be admissible, such as the one rescaling the standard Normal by δ, i.e. transforming the N(0,1) into N(0, δ). It could be interpreted as a shock to the innovation volatility, i.e. a volatility impulse.

A further discussion of the extended definition of shocks is beyond the scope of this paper. Consequently, we focus below on the drift impulses to gaussian innovations.

3.2 Nonlinear AR(1) process

i) **The dynamics**.

As an illustration we consider a dynamic NLAR(1) model defined by:

$$Y_t = g(Y_{t-1}; \epsilon_t) = g^{*(1)}(Y_{t-1}; \epsilon_t), \text{ say,} \tag{3.4}$$

where (ϵ_t) is a standard Gaussian white noise and g an invertible function with respect to ϵ. By recursion we find the expression of Y_t as a function of Y_0 and the innovations ${}_t\epsilon_1 = (\epsilon_1, ..., \epsilon_t)$:

$$Y_t = g^{*(t)}[Y_0, {}_t\epsilon_1], \tag{3.5}$$

where $g^{*(t)}$ is recursively defined by:

$$\begin{aligned} g^{*(t)}[Y_0, {}_t\epsilon_1] &= g\{g^{*(t-1)}[Y_0, {}_{t-1}\epsilon_1], \epsilon_t\} && (3.6) \\ &= g^{*(t-1)}\{g[Y_0, \epsilon_1], {}_t\epsilon_2\}. && (3.7) \end{aligned}$$

The equality (3.6) implies:

$$\frac{\partial g^{*(t)}}{\partial y}[Y_0, {}_t\epsilon_1] = \frac{\partial g}{\partial y}[Y_{t-1}, {}_t\epsilon]\frac{\partial g^{*(t-1)}}{\partial y}[Y_0, {}_{t-1}\epsilon_1],$$

and by the chain rule:

$$\frac{\partial g^{*(t)}}{\partial y}[Y_0, {}_t\epsilon_1] = \prod_{\tau=1}^{t} \frac{\partial g}{\partial y}[Y_{\tau-1}, \epsilon_\tau]. \tag{3.8}$$

ii) **Local impact of a transitory shock at date 1**.

Let us consider a small transitory shock $\delta_1 = \delta$. We get:

$$\begin{aligned} Y_t(\delta) &= g^{*(t)}[Y_0, {}_t\epsilon_1 + \delta_1^t] \\ &= g^{*(t-1)}[g(Y_0, \epsilon_1 + \delta), {}_t\epsilon_2], \quad \text{from (5.7),} \\ &\approx g^{*(t-1)}[Y_1, {}_t\epsilon_2] + \frac{\partial g^{*(t-1)}}{\partial y}[Y_1, {}_t\epsilon_2]\frac{\partial g}{\partial y}[Y_{\tau-1}, \epsilon_\tau]\delta \\ &= Y_t + \prod_{\tau=1}^{t} \frac{\partial g}{\partial y}[Y_{\tau-1}, \epsilon_\tau]\delta, \quad \text{from (3.8).} \end{aligned}$$

For a nonlinear transformation of the process, H say, we get:

$$H[Y_t(\delta)] \approx H(Y_t) + \frac{dH}{dy}(Y_t) \prod_{\tau=1}^{t} \frac{\partial g}{\partial y}[Y_{\tau-1}, \epsilon_\tau]\delta. \tag{3.9}$$

Therefore the infinitesimal effect of the transitory shock on the expectation of $H(Y_t)$ conditional on Y_0 is:

$$\frac{1}{\delta} E_0\{H[Y_t(\delta)] - H(Y_t)\} = E_0 \left[\frac{dH}{dy}(Y_t) \prod_{\tau=1}^{t} \frac{\partial g}{\partial y}[Y_{\tau-1}, \epsilon_\tau] \right]. \tag{3.10}$$

It is interesting to consider the long run impact of a transitory shock, i.e. the behaviour of either $\frac{1}{\delta}[Y_t(\delta) - Y_t]$, or $\frac{1}{\delta}E_0[Y_t(\delta) - Y_t]$, when t tends to infinity. Indeed Nelson (1990) [see also Bougerol, Picard (1992)] has shown in the framework of GARCH models that responses to shocks can be significantly different. More precisely, if the process is nonlinearly regular, then it has a short memory (see Definition 3), and in consequence the shock effect dissipates in the long run. Therefore asymptotically, the effect of a temporary shock to Y_t dies out to zero. For large t this effect is:

$$\begin{aligned} &\prod_{\tau=1}^{t} \frac{\partial g}{\partial y}[Y_{\tau-1}, \epsilon_\tau] \\ &= \exp\left\{ \sum_{\tau=1}^{t} \left[\frac{\log \partial g}{\partial y}(Y_{t-1}, \epsilon_\tau) \right] \right\} \\ &\sim \exp\left[tE \log \frac{\partial g}{\partial y}(Y_{\tau-1}, \epsilon_\tau) \right], \end{aligned}$$

and tends to zero if $E \log \frac{\partial g}{\partial y}(Y_{\tau-1}, \epsilon_\tau) < 0$, which is a necessary condition for a strongly stationary regular process, since $E \log \frac{\partial g}{\partial y}(Y_{\tau-1}, \epsilon_\tau)$ is the Liapunov exponent of the dynamic system [Oseledec (1968)].

However, by taking the expectation and using the convexity inequality, we get:

$$\begin{aligned} &E\left[\prod_{\tau=1}^{t} \frac{\partial g}{\partial y}(Y_{\tau-1}, \epsilon_\tau) \right] \\ &= E\left[\exp \sum_{\tau=1}^{t} \log \frac{\partial g}{\partial y}(Y_{\tau-1}, \epsilon_\tau) \right] \\ &\geq \exp\left[E \sum_{\tau=1}^{t} \log \frac{\partial g}{\partial y}(Y_{\tau-1}, \epsilon_\tau) \right] \\ &\approx \exp\left[t\, E \log \frac{\partial g}{\partial y}(Y_{\tau-1}, \epsilon_\tau) \right], \end{aligned}$$

and the stationarity condition $E\left[\log \frac{\partial g}{\partial y}(Y_{\tau-1}, \epsilon_\tau)\right] < 0$ does not necessarily imply that the impulse response vanishes in average for large t.

iii) **Linear AR(1) model with a random autoregressive coefficient.**

Let us consider the dynamic bilinear model [Tong (1993), p.8]:

$$Y_t = (a + b\epsilon_t)Y_{t-1} + \epsilon_t, \quad t \text{ varying},$$

where (ϵ_t) is a standard Gaussian white noise. We can explicitly compute the effect of a transitory shock hitting at date 1. The perturbed path is such that:

$$Y_t^D = (a + b\epsilon_t)Y_{t-1}^D + \epsilon_t, \quad \forall t \geq 2.$$

We see that:

$$\begin{aligned} \Delta Y_t = Y_t^D - Y_t &= (a + b\epsilon_t)\Delta Y_{t-1} \\ &= \prod_{\tau=2}^{t}(a + b\epsilon_\tau)(1 + bY_0)(\delta\epsilon_1). \end{aligned}$$

This model satisfies the property of shock linearity because the effect of the shock is a linear function of $\delta\epsilon_1$. Moreover we know from Bougerol, Picard (1992) that, for large t, the coefficient:

$$\begin{aligned} &\prod_{\tau=2}^{t}(a + b\epsilon_\tau)(1 + bY_0) \\ &= \exp\left\{(t-1)\frac{1}{t-1}\sum_{\tau=2}^{t}\log(a + b\epsilon_\tau)\right\}(1 + bY_0) \\ &\sim \exp[(t-1)E\log(a + b\epsilon_\tau)](1 + bY_0) \end{aligned}$$

tends to zero if and only if $E\log(a + b\epsilon_\tau) < 0$. However the effect of the shock on the expectation of Y_t is :

$$E[\Delta Y_t | Y_0] = a^{t-1}(1 + bY_0)\delta\epsilon_1.$$

This average effect tends to zero if $|a| < 1$, which is a more stringent condition than the negativity of $E\log(a + b\epsilon_\tau)$, due to the convexity inequality.

Example 3.1: The ACD(1,1) introduced in subsection (2.3) belongs to this class of models. Indeed, the expected durations satisfy:

$$\Psi_t = c + (\alpha g(\epsilon_{t-1}) + \beta)\Psi_{t-1},$$

and the impact of a transitory shock is such that :

$$\Delta\Psi_t = (\alpha g(\epsilon_{t-1}) + \beta)\Delta\Psi_{t-1}, \text{ for } t \geq 3.$$

This effect asymptotically vanishes in average if

$$|E(\alpha g(\epsilon) + \beta)| = |\alpha + \beta| < 1,$$

and vanishes path by path if the Liapunov exponent is negative $E\log(\alpha g(\epsilon) + \beta) < 0$. These dissipation patterns depend on α, β and on the distribution of standardized durations. For instance, if $g(\epsilon)$ has an exponential distribution, the Liapunov exponent is

$$\int_0^{\infty} \log(\alpha x + \beta) \exp -x dx = \log\beta + \exp(\beta/\alpha)E_1(\beta/\alpha),$$

where $E_1(x) = \int_x^{\infty} \frac{\exp -t}{t} dt$ is the exponential integral [see Abramowitz, Stegun (1964), formula 5.1.1, page 228]. We plot in Figure 5.1 the set of all points (α, β) for which the Liapunov exponent is zero.

Insert Figure 5.1 Liapunov Exponent Frontier

Below this frontier, all pairs of (α, β) coordinates are associated with negative Liapunov exponents, while the coordinates (α, β) above it are associated with its positive values. We find that the frontier is decreasing from $(\alpha = 0.0, \beta = 1.0)$ down to $(\alpha \approx 1.4, \beta \approx 0.1)$. For β's greater than 1.4 the frontier approches asymptotically the α axis.

3.3 Simulation results

In this section we illustrate the computation and the analysis of impulse response functions using as examples the ACD(1,1) model and a factor model with a specific form of nonlinear temporal dependence.

i) **The autoregressive conditional duration (ACD) model.**

We consider the ACD(1,1) model of subsection (2.3) with an exponential distribution of the standardized durations. The initial values have been fixed to $\epsilon_0 = 0.0, y_0 = 2.0$. We perform two experiments involving two sets of parameter values:

· Experiment 1: $c = 1,\ \alpha = 0.3,\ \beta = 0.2$,

· Experiment 2: $c = 1,\ \alpha = 0.4,\ \beta = 0.64$.

In the first experiment the shock effect asymptotically vanishes in the mean and path by path, whereas in the second experiment we observe a different outcome. We consider transitory shocks

δ occurring at date 1, and taking values $\delta = -1, -0.9, ...0.9, 1$, with the benchmark corresponding to $\delta = 0.0$. The maximal horizon is $H = 10$.

We display in Figure 5.2 the joint simulated paths for the benchmark and two perturbed series with $\delta = +/-1$, for both experiments. The effects of shocks quickly dissipate in Experiment 1 whereas they are more persistent in Experiment 2 although they also vanish asymptotically.

Insert Figure 5.2: Simulated Paths.

We plot in Figure 5.3 the (marginal) distribution of $Y_t(\delta)$ for horizon $t = 3$, and $\delta = -1, 0, +1$, computed conditionally on the information available at date 0.

Insert Figure 5.3: Marginal Distribution at Horizon 3.

We note that the shocks have an effect on the means and tails of the distribution. These effects can be evaluated by considering two summary statistics:

- the mean deviation $EY_t(\delta) - EY_t$,

- the variance of the deviation from benchmark: $V(Y_t(\delta) - Y_t)$,

for different horizons $t = 1, ..., 10$ and different values of transitory shocks. They are shown in Figures 5.4 and 5.5 for the experiment 1, and in Figures 5.6 and 5.7 for the experiment 2.

Insert Figure 5.4: Mean Deviation from the Benchmark.

Insert Figure 5.5: Variance of the Deviation from Benchmark.

Insert Figure 5.6: Mean Deviation from the Benchmark.

Insert Figure 5.7: Variance of the Deviation from Benchmark.

We observe explosive patterns of the averaged effects of shocks in the second experiment, although we found earlier no explosive paths. The patterns of responses associated to different shocks are similar due to the simple formula of the deviation from benchmark for the ACD(1,1) model. Indeed it is easy to see that :

$$Y_t(\delta) - Y_t = \alpha \prod_{\tau=3}^{t} \left[\alpha g(\epsilon_\tau) + \beta \frac{g(\epsilon_\tau)}{g(\epsilon_{\tau-1})} \right] g(\epsilon_2) \left[c + \left(\alpha + \frac{\beta}{g(\epsilon_0)} \right) y_0 \right] [g(\epsilon_1 + \delta) - g(\epsilon_1)],$$

which implies :

$$E[Y_t(\delta) - Y_t] = A_t E[g(\epsilon_1 + \delta) - g(\epsilon_1)],$$

where A_t is a positive number depending on the horizon. For the same reason we get:

$$\begin{aligned} V[Y_t(\delta) - Y_t] &= E[Y_t(\delta) - Y_t]^2 - (E[Y_t(\delta) - Y_t])^2 \\ &= B_t E[g(\epsilon_1 + \delta) - g(\epsilon_1)]^2 - A_t^2 \left(E[g(\epsilon_1 + \delta) - g(\epsilon_1)]\right)^2 . \end{aligned}$$

The response function depends on both the magnitude of the shock and the horizon. To clarify the dependence with respect to the shock size we reproduce in Figures 5.8-5.9 the Figures 5.4-5.5, with δ measured on the x-axis. We find that the response function is convex for the mean deviation with a stronger convexity asssociated to negative shocks. In particular the properties of symmetry and linearity of linear impulse responses are not satisfied. The variance function displays an asymmetric effect of positive and negative shocks.

Insert Figure 5.8: Mean Deviation from the Benchmark.

Insert Figure 5.9: Variance of the Deviation from Benchmark.

An impulse response analysis based only on the differences $Y_t(\delta) - Y_t$ can be misleading since it is not informative about the shock effect with respect to the mean of the transformed series. It is more insightful to consider the joint bivariate distribution of $Y_t(\delta), Y_t$ and examine how it depends on the magnitude of the shock and on the horizon. The corresponding scatterplots are given in Figure 5.10 for $\delta = +1, -1$, and horizon 3.

Insert Figure 5.10: Scatterplot at Horizon 3.

ii)**A model with linear autoregressive factors.**

We introduce a Gaussian AR(1) model:

$$Z_t = \rho Z_{t-1} + \epsilon_t, \quad t \quad \text{varying}, \tag{3.11}$$

and the process of interest defined by:

$$Y_t = a(\epsilon_t, Z_t), \quad t \quad \text{varying}, \tag{3.12}$$

where a is a given function. This process is generated by two underlying factors ϵ_t, Z_t based on the same Gaussian white noise. For a transitory shock δ hitting the process at date 1, we get:

$$Y_t(\delta) = a(\epsilon_t, Z_t + \rho^{t-1}\delta), \quad t \geq 2,$$

and the joint distribution of $[Y_t, Y_t(\delta)]$ can easily be deduced from the joint Gaussian distribution of ϵ_t, Z_t. The effect of the shock depends on the nonlinear transformation a.

Let us first consider the function:

$$a(\epsilon_t, Z_t) = sign(\epsilon_t) \exp Z_t. \tag{3.13}$$

We can see that: $Y_t(\delta) = Y_t \exp(\rho^{t-1}\delta)$, and the joint distribution of $[Y_t, Y_t(\delta)]$ is degenerate so that its support is a line passing through the origin. The shock has no effect on the sign of Y_t, whereas it has a multiplicative effect on its absolute value.

Less extreme examples corresponding to the functions: $a(\epsilon_t, Z_t) = \epsilon_t Z_t$ and $a(\epsilon_t, Z_t) = Z_t/\epsilon_t$, respectively are illustrated in Figures 5.11-5.11. The values of the parameters are: $\rho = 0.9$, $\delta = +1, -1$, and the horizon is $t = 4$.

Insert Figure 5.11 Scatterplot a = epsilon*Z

Insert Figure 5.12 Scatterplot a = Z / epsilon

These examples show that in some cases, the joint distributions of $[Y_t, Y_t(\delta)]$ have complicated forms which are poorly described by comparing only the expectations or the variances. Therefore an impulse response analysis may eventually require more detailed summary statistics.

4 Value at Risk for a dynamic financial strategy

The specification and estimation of dynamic models for economic or financial series of is often a preliminary step in the decision making about dynamic strategies, such as portfolio hedging and market interventions. In such a framework we are more interested in studying the consequences of shocks on the outcomes of a dynamic strategy rather than in examining their effects on the series itself. Even for time series with simple dynamics, the choice of a strategy is often determined by solving either a myopic or an intertemporal optimization problem, both yielding nonlinear outcomes. In that case, impulse responses to the nonlinear solution do not admit analytical expressions. To circumvent this difficulty, it is possible in practice to linearize the outcomes by replacing the objective function in the optimization problem by a quadratic objective function. However, this approach entails a bias of the outcome and, in consequence, of the impulse response function as well. Instead, it seems therefore preferable to approximate directly the complicated, nonlinear impulse responses by simulations. In this section, we discuss the structural interpretation of impulse response functions to the Value at Risk [VaR] employed in finance to measure and control the risks associated with a portfolio. In the first subsection we recall the standard definition of the VaR, and extend this definition to a dynamic strategy of risk assessment in the second subsection.

4.1 Definition of the VaR

We consider at date T a portfolio including the quantities $a_{0,T}$ and a_T of a riskfree asset and various risky assets, respectively. We denote by $y_{0,T+h}, y_{T+h},\ h = 0, 1, ..., H$ the future values of the assets. According to regulation and monitoring rules the VaR is computed for portfolios, whose allocations are constrained to be fixed in the future [the so-called crystallization of the portfolio]. The future portfolio values are:

$$W_{T+h} = a_{0,T} y_{0,T+h} + a'_T y_{T+h},\ \ h = 0, 1, ..., H. \tag{4.1}$$

These values are random conditional on the information available at time T.

The Value at Risk of this portfolio evaluated at time T, for the horizon H and the critical value $\alpha \in [0, 1]$ is the quantity VaR [T, H, α] defined by:

$$P_T[W_{T+H} > -VaR\ \ [T, H, \alpha]] = 1 - \alpha, \tag{4.2}$$

where P_T is the conditional distribution of future prices. The VaR may be used to determine the minimal capital requirement to ensure that the total wealth, including the portfolio value and the capital requirement, remains positive with a sufficiently large probability. More precisely, let us assume that the capital requirement can be invested at a rate providing a zero coupon price $B[T, H]$ for horizon H. Consequently, the VaR determined capital requirement would be:

$$R[T, H, \alpha] = \frac{VaR\ \ [T, H, \alpha]}{B[T, H]}. \tag{4.3}$$

Despite that it would be optimal to measure risk ex ante, it is in practice usually measured ex post. The reason for this is the belief that the conditional distribution P_T can be well approximated by the historical distribution estimated from recent past data[see Morgan (1994)]. However when a dynamic model of asset prices is available, an ex-ante computation of the VaR can be performed by simulation. For instance, let us suppose that prices follow a (nonlinear) AR(1) model:

$$y_t = g(y_{t-1}, \epsilon_t),\ \ t\ \text{ varying}, \tag{4.4}$$

where g is a given function, and $y_{0,T+h} = 1,\ \forall h = 1, ..., H$. We can simulate the future risky asset prices given the current value y_T by:

$$y^s_{T+h} = g(y^s_{T+h-1}, \epsilon^s_{T+h}),\ \ h = 1, ..., H,\ s = 1, ..., S, \tag{4.5}$$

where $y^s_T = y_T$, and we deduce the simulated future values of the portfolio by:

$$W^s_{T+h} = W_T + a'_T(y^s_{T+h} - y_T), \quad h = 1, ..., H, \ s = 1, ..., S. \tag{4.6}$$

Then the VaR can be computed from the empirical distribution of these simulated values W^s_{T+h}, $s = 1, ..., S$.

Let us now introduce a shock δ to the innovation at date $T + 1$. We can perform similar computations after replacing ϵ^s_{T+1} by $\epsilon^s_{T+1} + \delta$, and deduce the distribution of the future portfolio values $W^s_{T+h}(\delta)$, say, along with the associated Value at Risk: VaR [T, H, α; δ]. This will allow us to study the sensitivity of the VaR and of the capital requirement to a transitory shock δ.

4.2 Extension to dynamic strategies

The approach presented above has been proposed by the regulators for its simplicity, despite that in practice it yields suboptimal results. Indeed, it is based on the assumption of fixed future portfolio allocations, whereas the investors regularly update these allocations to take advantage of price movements. A typical example is a hedging portfolio for an european call. The portfolio is often updated at regular dates $T + h$, $h = 1, ..., H$ (say), with allocations determined by the deltas of the Black-Scholes formula. Even though the price evolution in the Black-Scholes model obeys linear dynamics, the deltas are complicated nonlinear functions of prices. Thus the risk is mainly reduced by implementing nonlinear adjustments.

Let us still assume the price of the riskfree asset $y_{0,T+h} = 1$, $h = 1, .., H$. The future values of a self-financed portfolio are:

$$W_{T+H}[a(.)] = W_T + \sum_{h=1}^{H} a_h(\underline{y_{T+h-1}})[y_{T+h} - y_{T+h-1}], \tag{4.7}$$

where $a_h(\underline{y_{T+h-1}})$ are the allocations in the risky assets considered at the h^{th} updating, which are path dependent in general.

For a given dynamic strategy $a(.) = [a_h(.), \ h = 1, ..., H]$, we can compute, like in the previous subsection, the simulated future portfolio values

$$W^s_{T+H}[a(.)] = W_T + \sum_{h=1}^{H} a_h(\underline{y^s_{T+h-1}})[y^s_{T+h} - y^s_{T+h-1}], \tag{4.8}$$

and deduce the VaR's with and without a transitory shock. These Values at Risk will depend on the selected dynamic strategy. Let us denote them by:

$$VaR \ [T, H, \alpha; \ a(.)] \ \text{ and } \ VaR \ [T, H, \alpha; \ a(.); \delta].$$

They can be compared to the Values at Risk evaluated for a portfolio with fixed allocations $a_h(.) = a_T$, $\forall h$:

$$VaR\ [T, H, \alpha] \text{ and } VaR\ [T, H, \alpha; \delta],$$

using the notation of subsection 6.1.

4.3 Application

As an illustration we consider porfolios invested in one month T-Bill (i.e. the riskfree asset) and the NYSE market index (i.e. the risky asset). These portfolios may be updated every week. The price of the riskfree asset is set to $y_{0,t} = 1$, whereas the net price of the risky asset is denoted by y_t. The dynamics of the associated net returns r_t has been estimated on weekly data for the period of July 1962 - December 1985 using a GARCH-M model [Chou (1988)]. The estimated model is:

$$\begin{array}{cccccc} r_t & = & \delta & h_{t-1}^2 & + & h_t\epsilon_t \\ & = & \underset{(3.28)}{4.56} & h_{t-1}^2 & + & h_t\epsilon_t, \end{array} \tag{4.9}$$

where:

$$\begin{array}{cccccccccc} h_t^2 & = & \alpha_0 & + & \alpha & h_{t-1}^2\epsilon_{t-1}^2 & + & \beta & h_{t-1}^2 \\ & = & \underset{(2.83)}{0.099 * 10^{-4}} & + & \underset{(7.15)}{0.151} & h_{t-1}^2\epsilon_{t-1}^2 & + & \underset{(38.64)}{0.834} & h_{t-1}^2, \end{array} \tag{4.10}$$

and $\epsilon_t \sim IIN(0,1)$. The t-ratios are given in brackets. Note that lagged volatility has been introduced into the conditional mean equation to avoid a time independent risk premium.

We consider arbitrage portfolios with a zero initial investment $W_0 = 0$ and a horizon $H = 52$ weeks corresponding to one year. The first portfolio is static with a mean-variance allocation [Markowitz (1952)] computed at the initial date and associated to a risk aversion coefficient $A = 1$ [4]. Its value at the terminal date is:

$$W_H = \frac{\delta h_{-1}^2}{h_0^2 y_0}(y_H - y_0). \tag{4.11}$$

The second portfolio is regularly updated every week using a time dependent mean-variance allocation and the same risk aversion coefficient. Its terminal value is:

$$W_H(a) = \sum_{j=1}^{H} \frac{\delta h_{j-1}^2}{h_j^2 y_{j-1}}(y_j - y_{j-1}). \tag{4.12}$$

Thus the complicated dynamic nonlinear effects are due to both the GARCH-M estimated dynamics of the net returns and the nonlinear relationship between the portfolio value of the self-financed portfolio and the return history.

[4] Due to zero investment the terminal values of the static and dynamic portfolios are simply multiplied by $1/A$ in the case of a risk aversion coefficient different from 1. This does not modify the patterns of their distributions.

Even though the succession of myopic allocations is not intertemporally optimal, the dynamic strategy is expected to be advantageous. However it is not clear whether it is preferable with respect to both the expected value and its variability. Moreover, one could fear that this more sophisticated strategy is less robust and more sensitive to external shocks. To analyze these issues we fix the initial value h_{-1}^2 of the volatility to its expectation estimated from the residuals of the last iteration step in (4.10), and the value y_0 to the last value from the sample. We first provide in Figure 6.1 the p.d.f. of the terminal values for the static and dynamic portfolio managements.

Insert Figure 6.1: Distribution of Terminal Values

We see that the means of the distributions are quite close whereas the tail of the distribution is heavier in the static case. Therefore the choice of an appropriate dynamic strategy may reduce the risk. This effect is confirmed when we examine the VaR computed under and without a transitory shock to the innovation at the initial date.

Insert Figure 6.2: VaR for Static and Dynamic Strategies

We find that the required amount of reserve is not only lower under the dynamic management, but it also is less sensitive to innovation shocks. Indeed the slope of the VaR is weaker for the dynamic strategy. We also observe asymmetric reactions of the VaR to positive and negative shocks. At first sight this result seems surprising. Indeed it is a common knowledge that major risks in portfolio management are mainly due to the trader's behavior, i.e. to the portfolio allocation updating, and not to the price uncertainty. The example above shows that a skilled allocation updating will diminish the risk.

5 Conclusions

This paper introduced the notion of nonlinear innovations for time series models which provides a unifying framework for the analysis of processes and their nonlinear transformations. The nonlinear impulse response analysis proposed in this paper covers both permanent and transitory shocks, and accounts for the asymmetric effects of shocks, shock nonlinearity and path dependency typical for nonlinear processes. The empirical part of the paper illustrated the impulse response analysis of the Value at Risk evaluation under a static and a dynamic portfolio management scheme. The identification problem of the nonlinear innovation and the difficulty in interpreting deterministic shocks show clearly that the impulse response function does not have a structural interpretation as a multiplier effect of a shock to economic policy. In fact the impulse response function arises rather as an useful tool for the sensitivity analysis of complicated nonlinear dynamics.

Figure 5.1: Liapunov Exponent = 0 Frontier

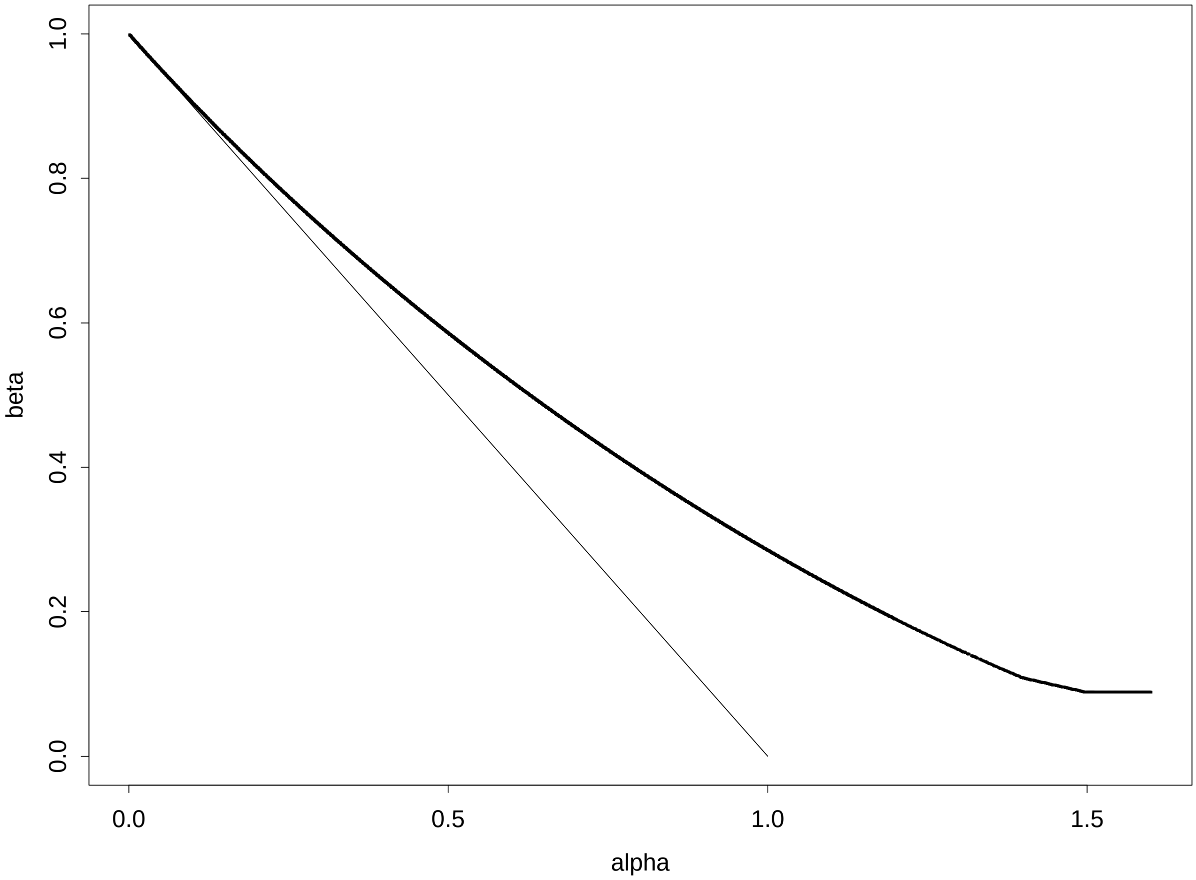


Figure 5.2: Simulated Paths

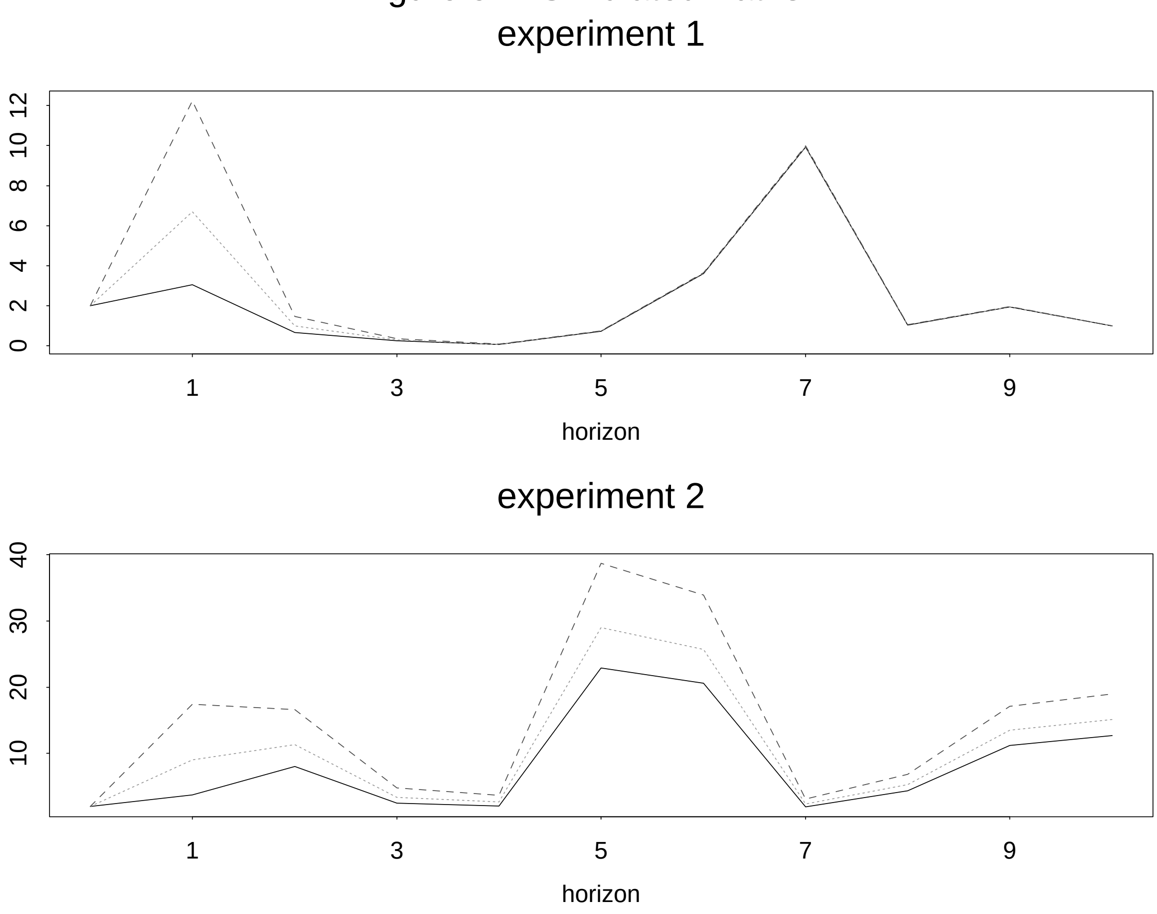


Figure 5.3: Marginal distributions at horizon 3

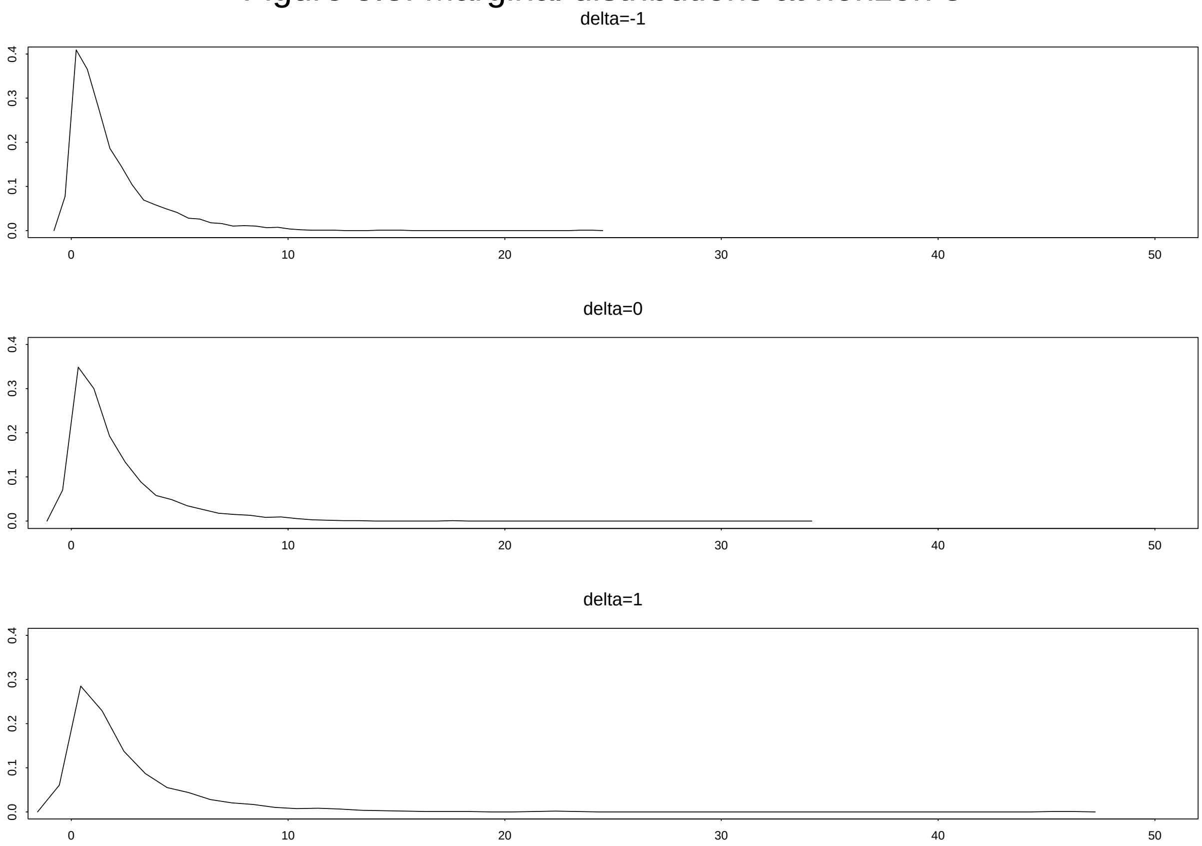


Figure 5.4: Mean Deviation from the Benchmark

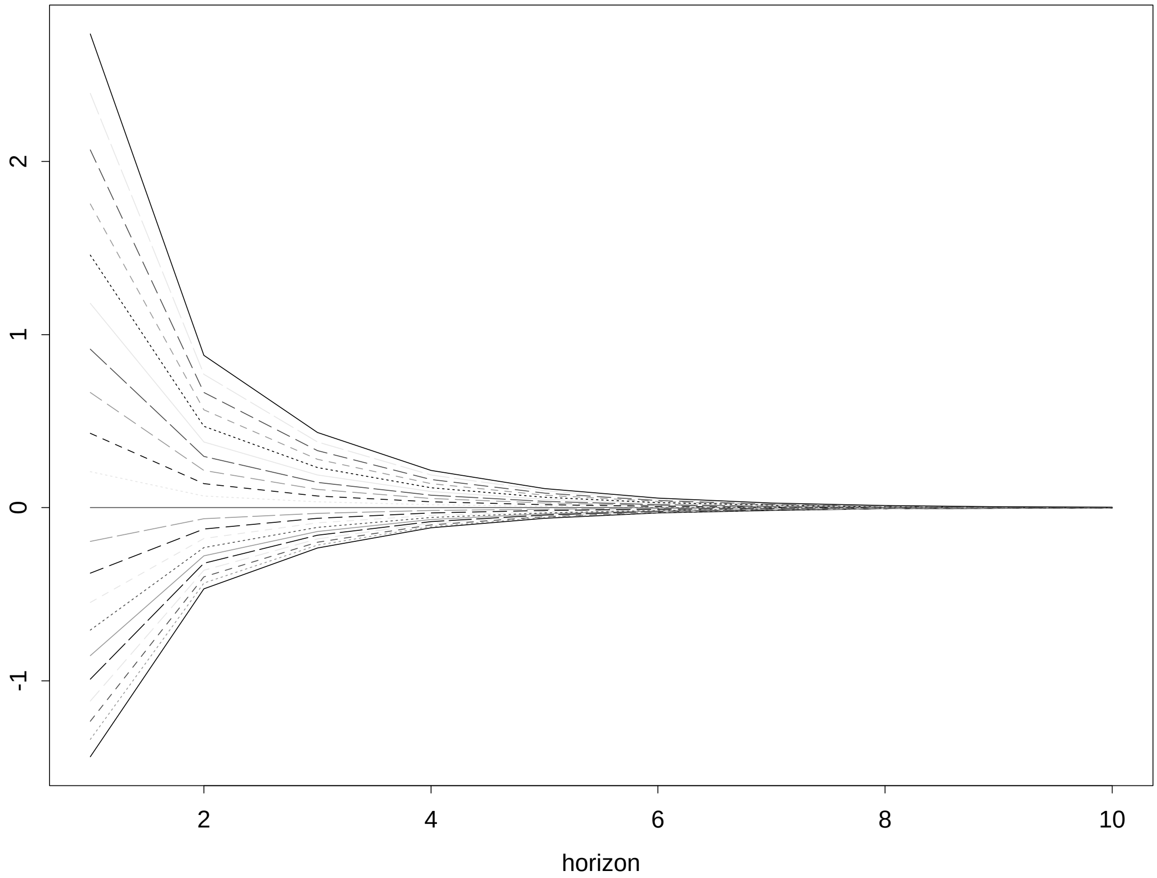


Figure 5.5: Variance of the Deviation from Benchmark

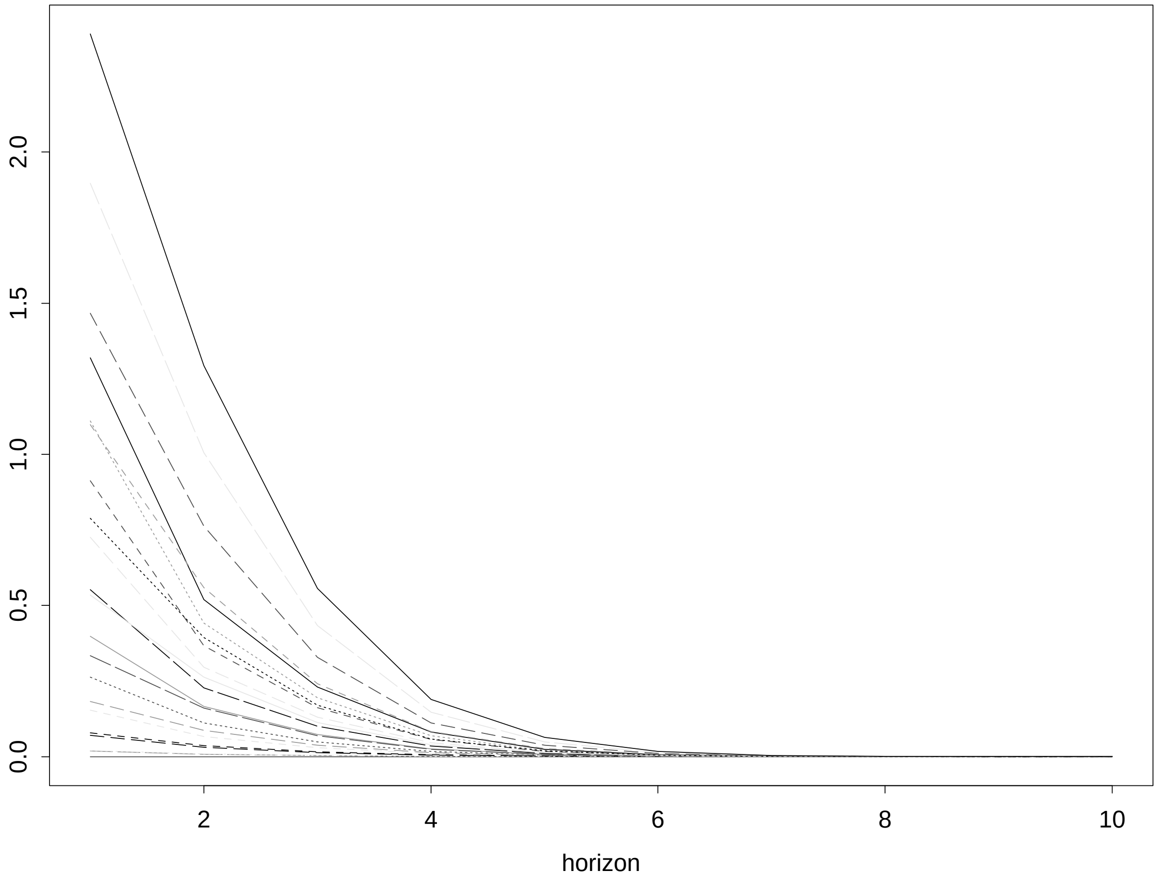


Figure 5.6: Mean Deviation from the Benchmark

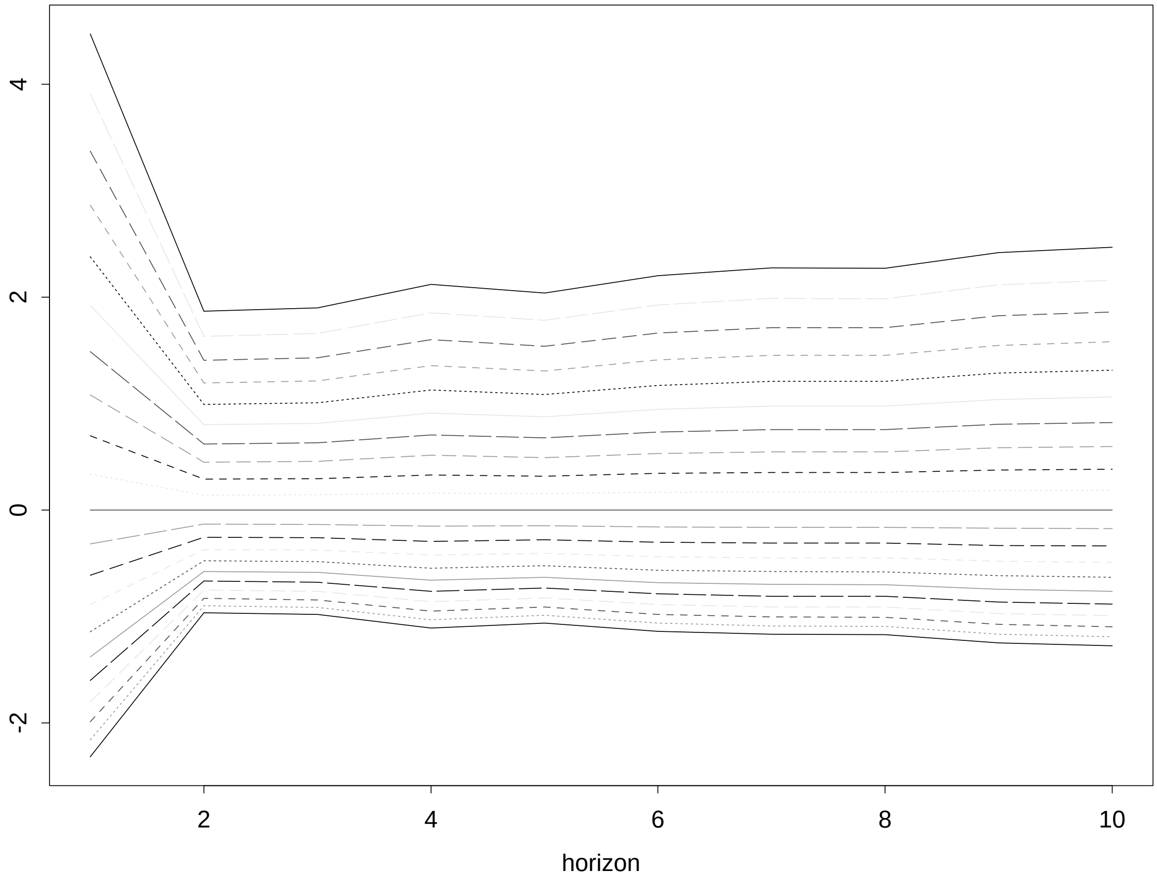


Figure 5.7: Variance of the Deviation from Benchmark

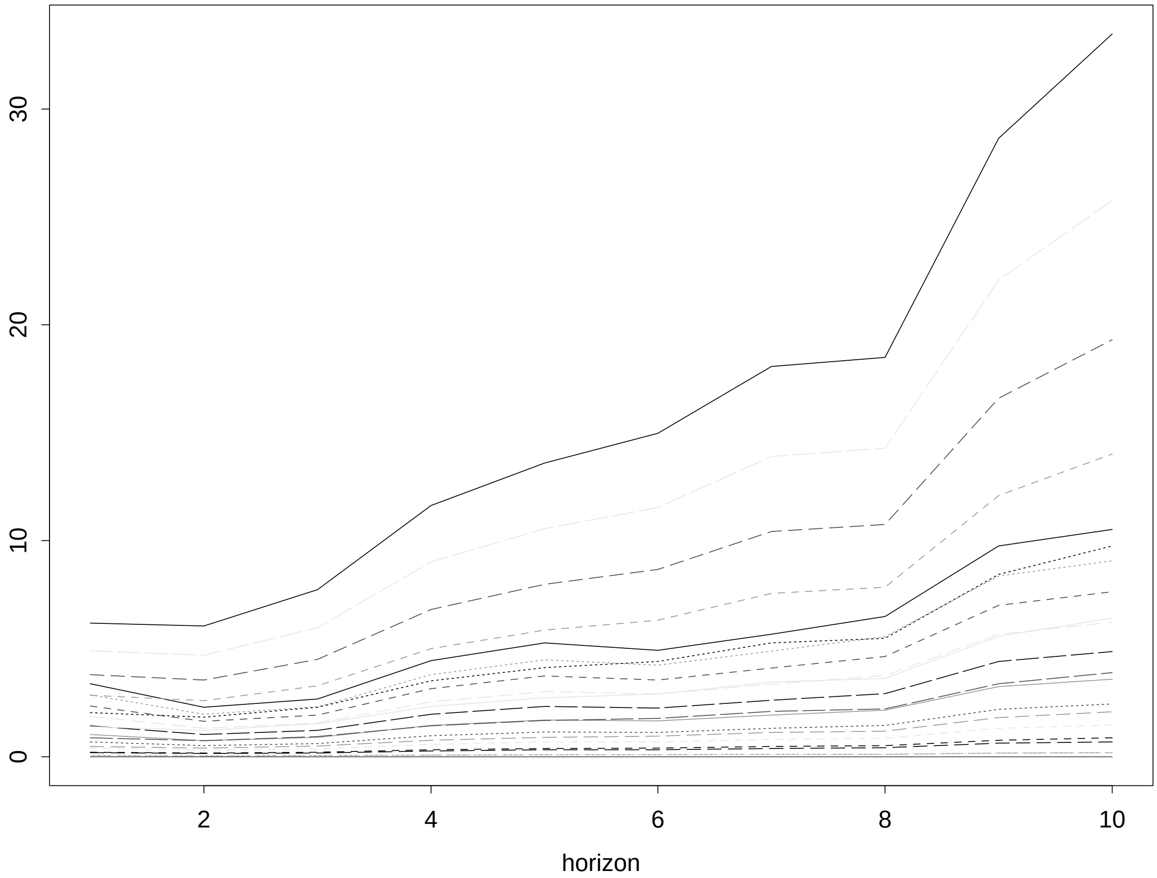


Figure 5.8: Mean Deviation from the Benchmark

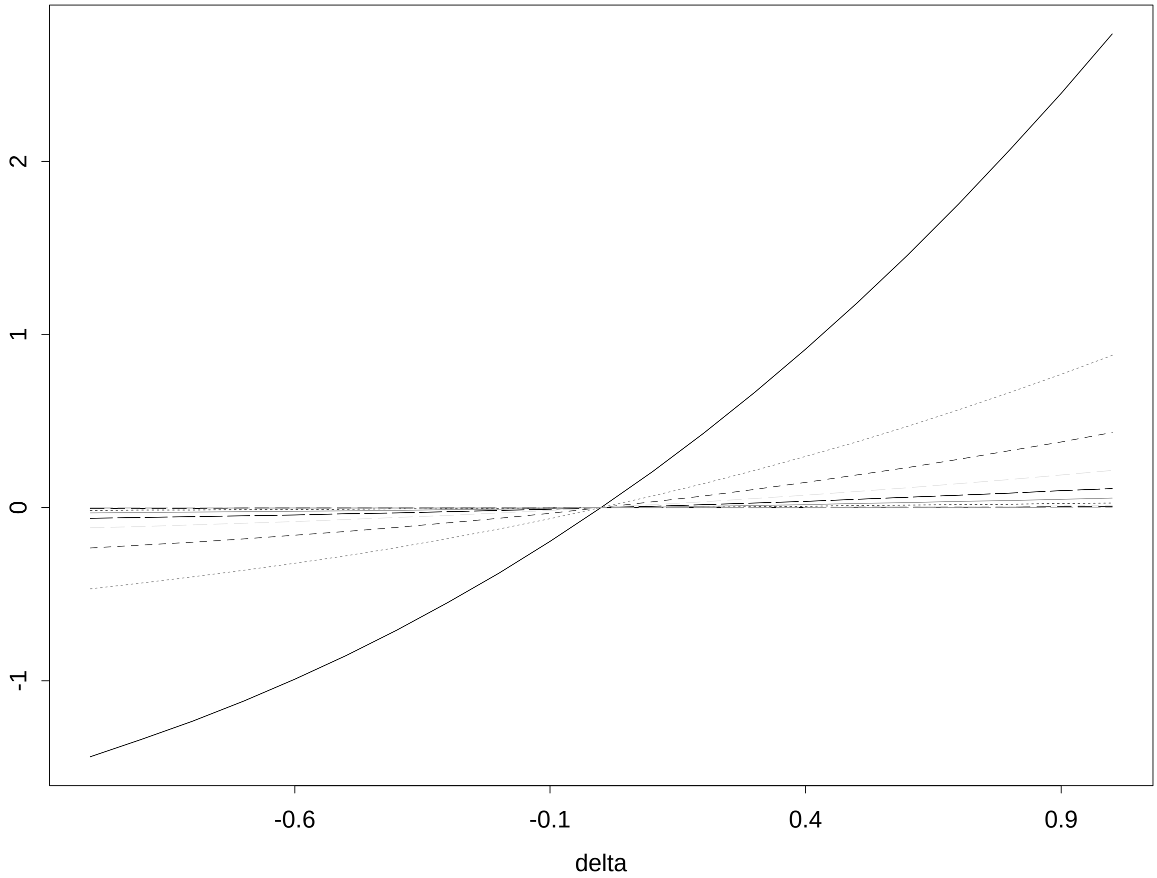


Figure 5.9: Variance of the Deviation from Benchmark

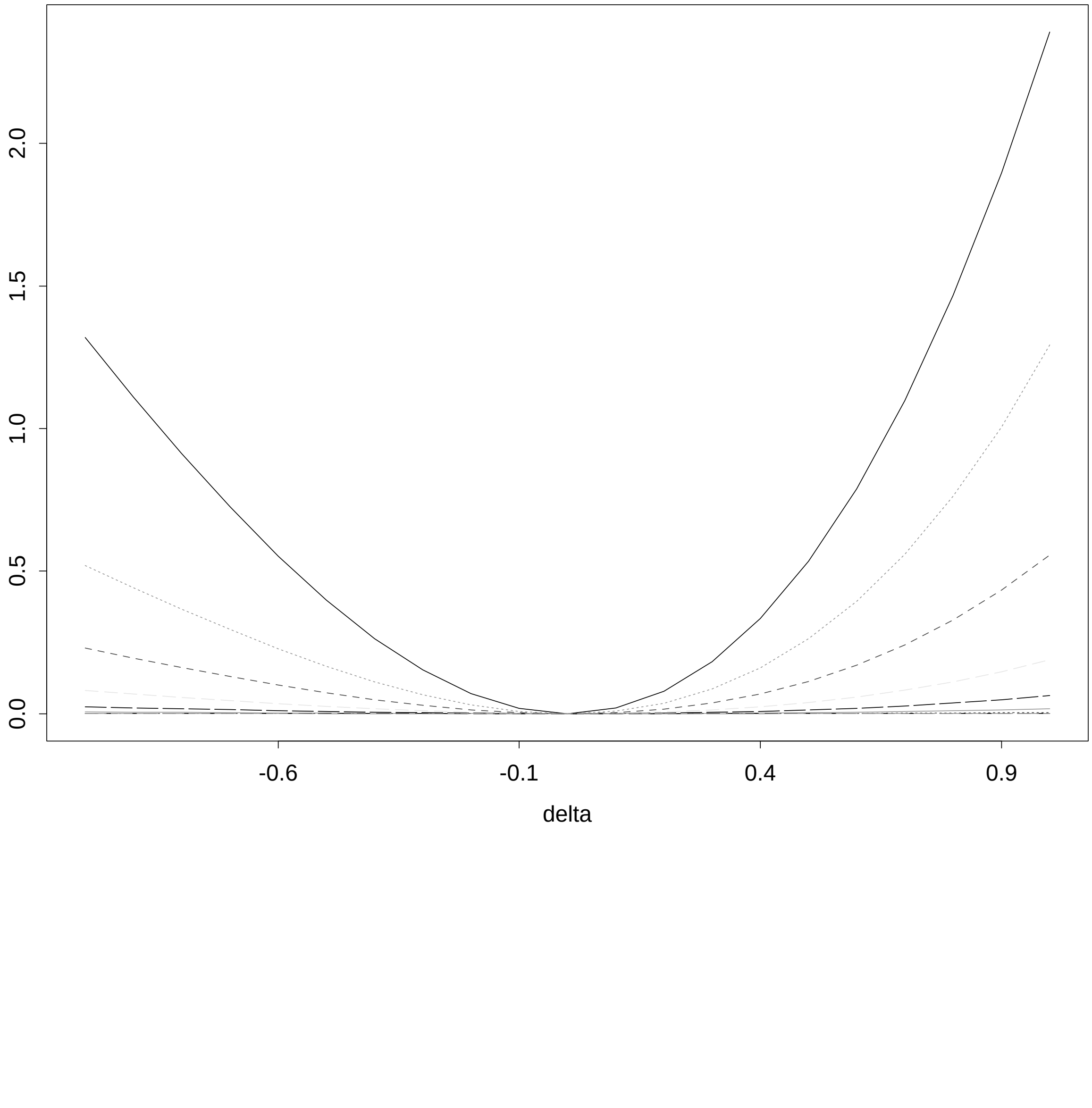
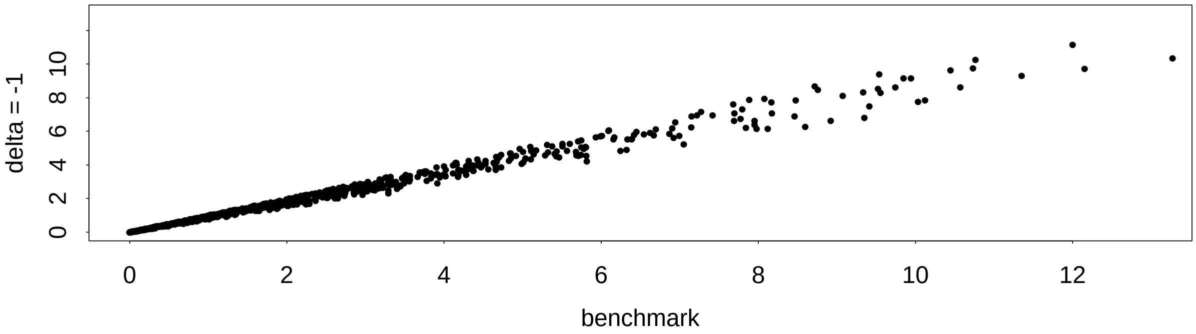


Figure 5.10: Scatterplot at Horizon 3



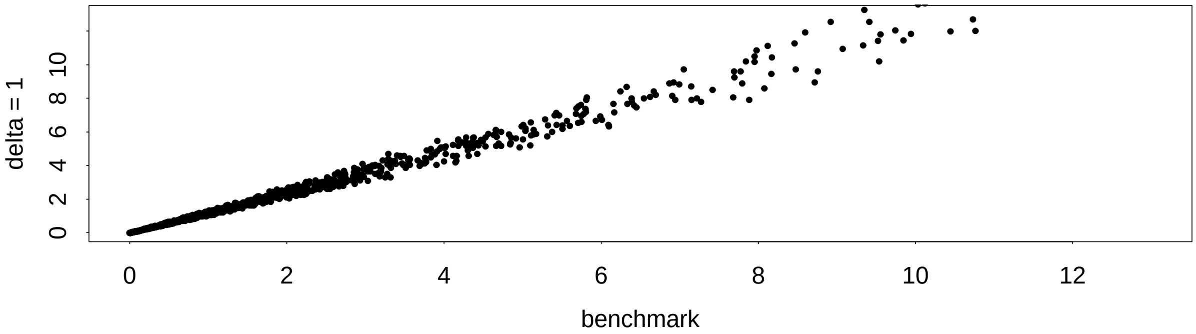


Figure 5.11: Scatterplot a = epsilon*z

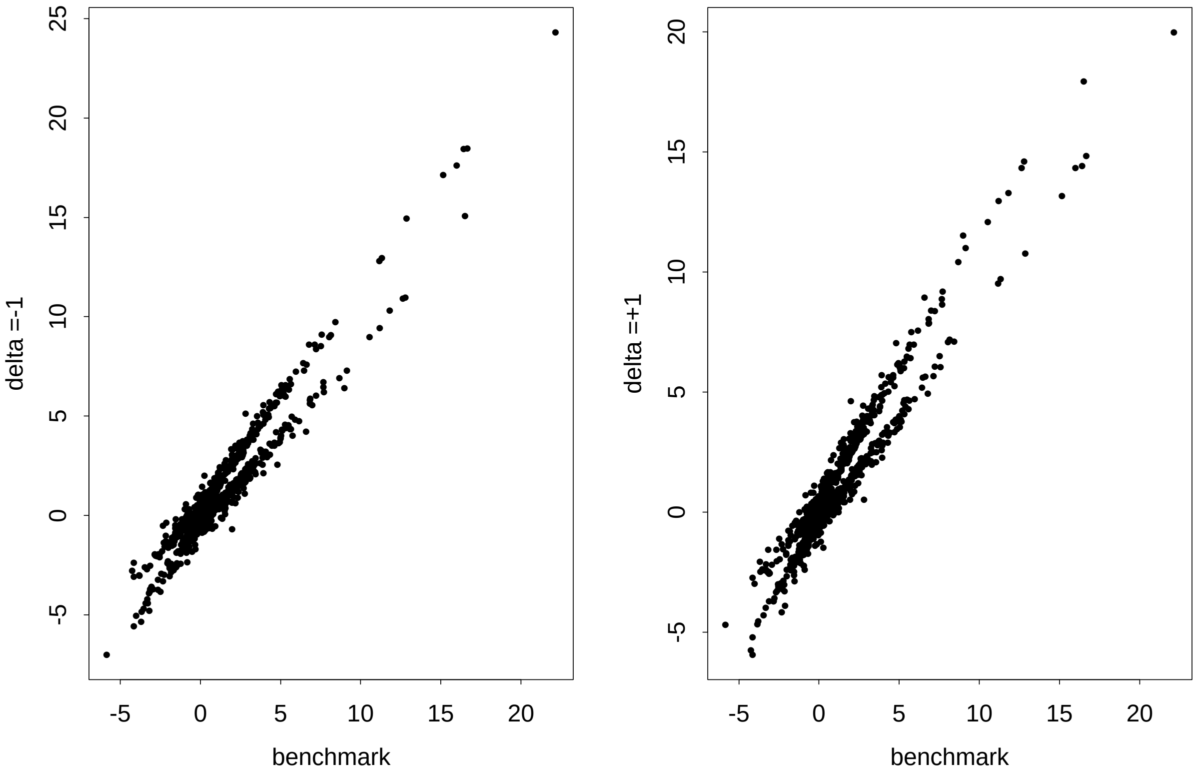


Figure 5.12: Scatterplot a = z / epsilon

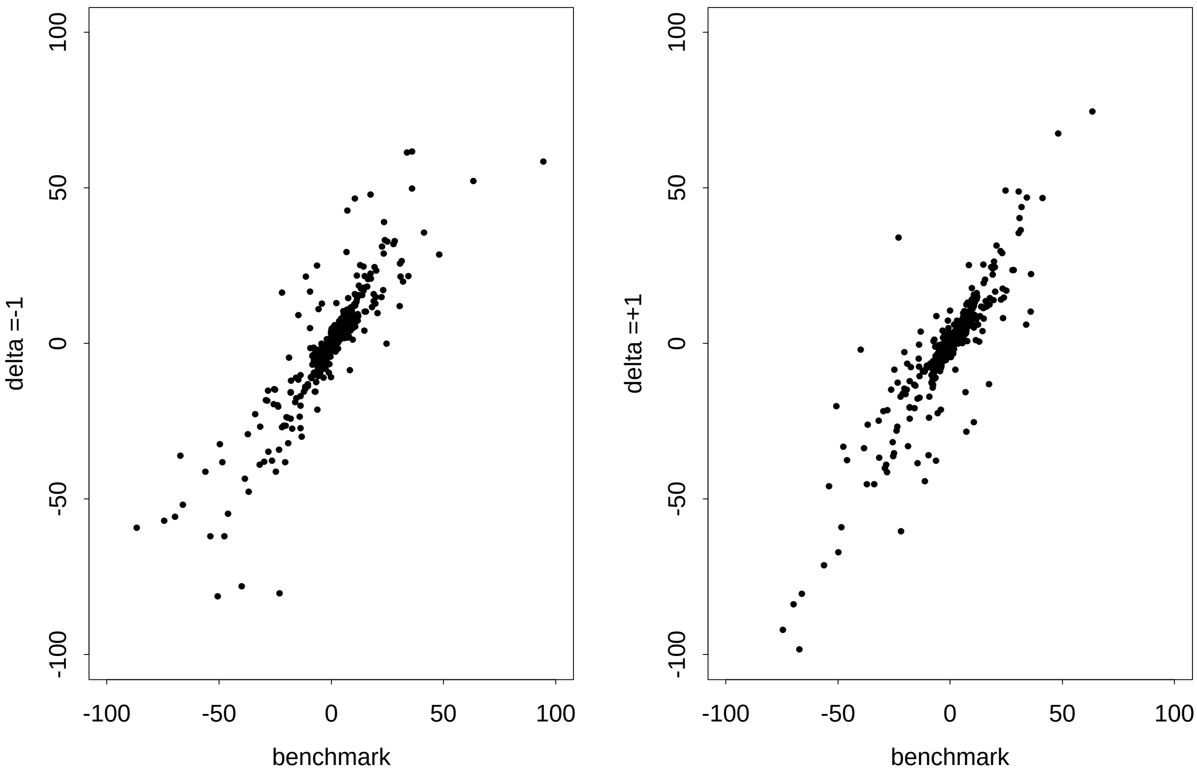


Figure 6.1: Distribution of Terminal Values

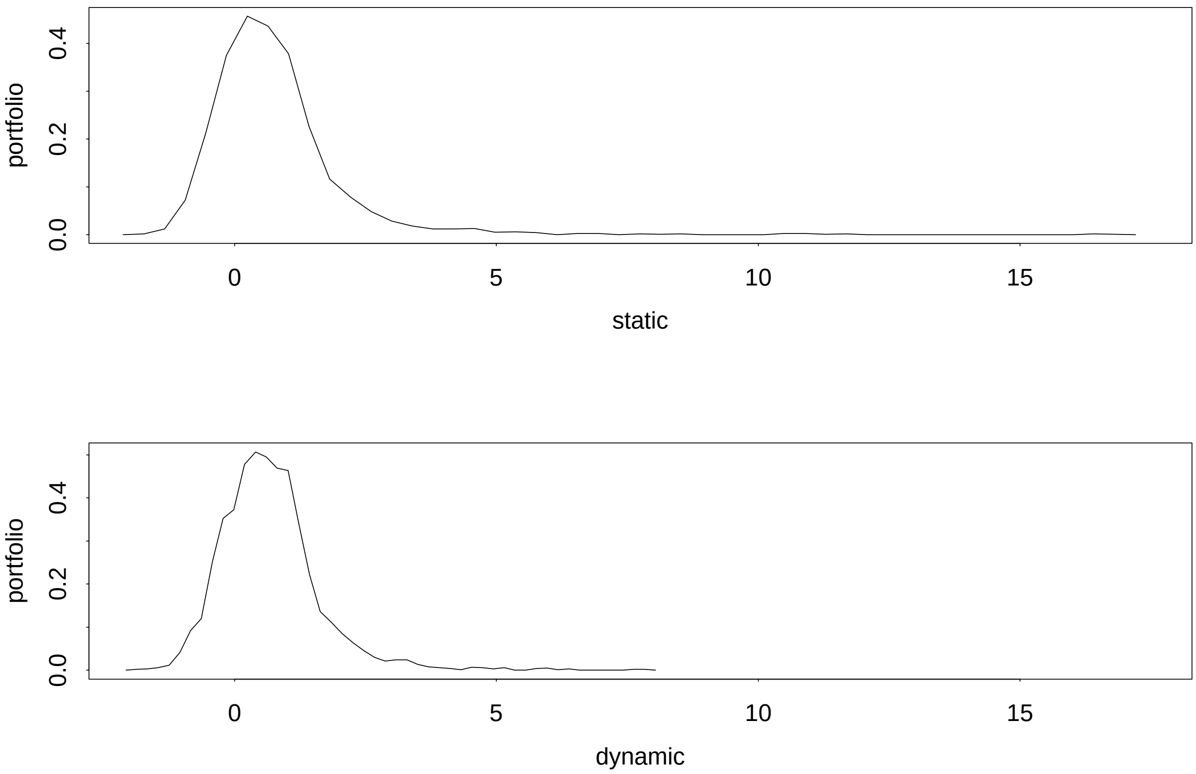
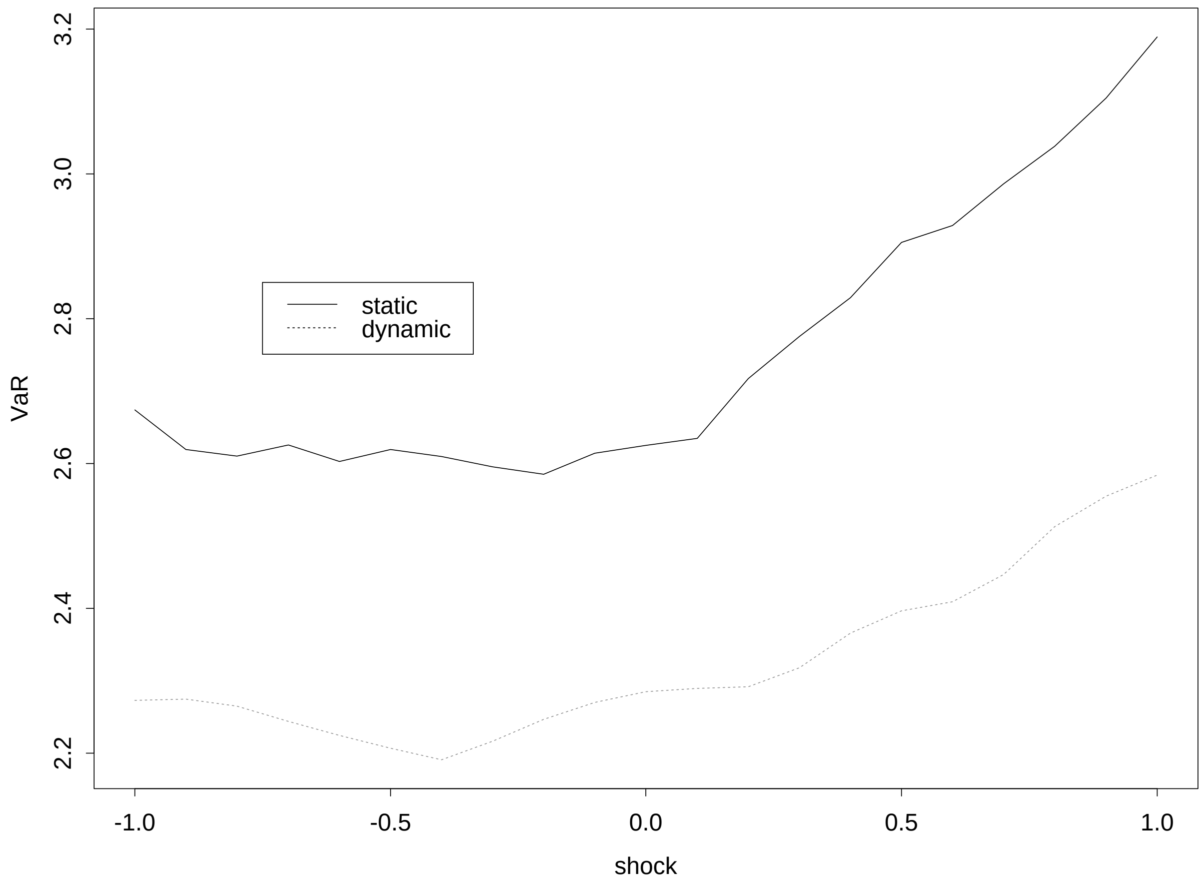


Figure 6.2: VaR for Static and Dynamic Strategies



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